On two conjectures that shaped the historiography of indeterminate analysis: Strachey and Chasles on Sanskrit sources

Ivahn Smadja

Univ Paris Diderot, Sorbonne Paris Cité, Laboratoire SPHERE, UMR 7219 CNRS, F-75205 Paris, France

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Abstract

This paper is part of a research project on the historiography of mathematical proof in ancient traditions. Its purpose is to shed light on the various ways in which nineteenth-century European scholars attempted to make sense of Sanskrit mathematical sources dealing with indeterminate analysis. Attention will be paid to the historical processes by which these different strands interwove into a cumulative historiography of the field. The focus is on two interpretive conjectures that shaped alternative readings of an evolving corpus of texts, with significantly different emphases and viewpoints.

The British scholar and East India Company servant Edward Strachey first identified a consistent algebraic theory in Bhāskara’s Bīja-ganita, which he translated from a seventeenth-century Persian manuscript. While reading his sources through the lens of the Euler–Lagrange theory of periodic continued fraction expansions for quadratic irrationals, he offered an insightful interpretation of the so-called cakra válā, or “cyclic method”. Two decades later, in the context of his investigations on the historiography of geometry, the French geometer Michel Chasles delved into Henry Thomas Colebrooke’s translations of Bhāskara and Brahmagupta, from the Sanskrit original, which had become authoritative all over Europe in the meantime. While working out an overall interpretation of Brahmagupta’s theory of quadrilaterals, Chasles incidentally spotted a geometrical construction which opened the way to a geometrical solution of the indeterminate equation $C x^2 \pm A = y^2$. He conjectured that this geometrical way may have been the Sanskrit path to indeterminate analysis. Furthermore, on the basis of textual reconstruction, he supplemented his rigorous interpretive conjecture with a more sweeping historical assumption about a possible transmission of this geometrical approach to algebra, from Sanskrit to European mathematics, through the Arabs and Fibonacci. Owing to further scholarship by Baldassare Boncompagni, Franz Woepcke and others, the wheat would be sorted out from the chaff.

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Résumé

Cet article s’inscrit dans un projet d’ensemble portant sur l’historiographie de la preuve mathématique dans les traditions anciennes. Il vise à mettre en lumière les différentes manières dont on a cherché au dix-neuvième siècle en Europe à faire sens de sources mathématiques sanskrites traitant d’analyse indéterminée. En retraçant les processus historiques par lesquels une historiographie cumulative du champ s’est constituée par l’entrecroisement de différentes approches, nous nous attacherez en particulier

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E-mail address: ivahn.smadja@univ-paris-diderot.fr.
à deux conjectures interprétatives qui ont façonné des lectures alternatives d’un corpus évolutif de textes, lesquels ont été envisagés dans des perspectives et selon des orientations significativement différentes.

Edward Strachey, érudit britannique au service de la Compagnie des Indes, fut le premier à discerner une théorie algébrique cohérente dans le *Bīja-ganita* de Bhāskara, dont il publia une traduction à partir d’un manuscrit persan du dix-septième siècle. En lisant ses sources à travers le prisme de la théorie des fractions continues périodiques due à Euler et Lagrange, il proposa une interprétation pénétrante de la méthode dite *cakravāla*, ou “méthode cyclique”. Deux décennies plus tard, dans le cadre de ses recherches sur l’histoire de la géométrie, le géomètre français Michel Chasles se plongea dans la lecture des œuvres mathématiques de Bhāskara et Brahmagupta, dans la traduction de Henry Thomas Colebrooke. Alors qu’il cherchait à élaborer une interprétation d’ensemble de la théorie des quadrilatères de Brahmagupta, Chasles remarqua incidemment une construction géométrique qui ouvrait la voie à une interprétation géométrique de l’équation indéterminée $Cx^2 + A = y^2$. Il émit alors la conjecture que cette approche géométrique pouvait avoir été la voie d’accès sanskrit à l’analyse indéterminée. En outre, sur la base d’une reconstruction textuelle, il se risqua à formuler une hypothèse historique hasardeuse au sujet de la possible transmission de cette approche géométrique de l’algèbre, des mathématiques sanskrites aux mathématiques européennes, en passant par les Arabes et Fibonacci. Les travaux ultérieurs de Baldassare Boncompagni, Franz Woepcke et d’autres, devaient par la suite permettre de séparer le bon grain de l’ivraie.

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### 1. Introduction

Recent debates among historians have shown a rising concern for the question of historiography, a call for reconsidering anew both its nature and prospects. A common awareness emerges that any worthwhile reflection on history writing should avoid two opposite fallacies, commonly labeled as the postmodernist and the empiricist fallacies. On the one hand, all historiography would run the risk to dissolve into mere fiction, self-pleading narrative, if not downright propaganda wholly determined by ideological biases. Postmodernist views spreading from literary criticism to the whole field of the humanities and the social sciences would presumably turn into utmost skepticism and hyper-relativism, and, by putting the very practice of historiography under severe strain, would eventually lead to dismissing the pursuit of objectivity as a forlorn illusion. Empiricist responses to these assaults, on the other hand, would rather clumsily try to rescue the cumulative character of historical knowledge, by unsophisticatedly vindicating the past as being granted in some realistic sense, albeit encrypted in the sources – a stance which was sometimes disparaged by its detractors as a “source fetishism”, or an “archive positivism”. Hence, the notion increasingly prevails that in devising future historiographies, historians should strive to steer a middle course between

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1. As representative of this trend, one may refer to the works of such historians as Thomas L. Haskell, Richard J. Evans, Mary Fulbrook, and others; cf. Haskell (1998), Evans (2000), Fulbrook (2002). Besides, on the part of philosophers, significant attempts are made at articulating a philosophy of historiography faithful to the historiographic practices of the historians; see for instance the work of Aviezer Tucker, John Zammito and others, cf. Tucker (2004), Zammito (2009).

2. Among the most influential contributions to a postmodernist approach to historiography, let us mention the work of Hayden White and Frank Ankersmit, cf. White (1973), Ankersmit (1989).

3. For an account of those late twentieth-century postmodernist challenges on historiography from the point of view of historians of historiography, see the monographs of Georg Igers and Ernst Breisach, cf. Igers (2012), Breisach (2003). There is a sense in which postmodernism cultivated a rhetoric of theoretical extremism which is hardly compatible with the enduringly provisional character of most historical knowledge. See Zammito (2009, p. 68): “Hyperbole pervades the rhetoric of poststructuralism and postmodernism . . . Historiographic practice was a prime target for these hyperbolic gambits.”

both extremes. However, the same should hold for any attempt at critically assessing past historiographies as well. In this respect, a distinction introduced by Leon J. Goldstein between what he dubbed, by appropriating well-known expressions with a twist of his own, the “superstructure” and the “infrastructure” of historiography, may prove useful for articulating an important point. Whereas the former is “the literary product of the historian’s work, the final form in which his conclusions are cast”, usually presented as a narrative, the latter is “that range of intellectual activities whereby the historical past is constituted in historical research; [which] involves treatment of evidence and thinking about evidence, and is preoccupied with the determination of what conception of the historical past makes best sense given the character of the evidence in hand”. For that reason, A. Tucker emphasizes, “the superstructure does not reflect the historiographic process of inquiry, the relation between historiography and evidence”, only the infrastructure does. As for the historiography of mathematics, the plot thickens. In that case, the very possibility of writing history implies by necessity that certain identifications with respect to ‘procedures’, ‘proofs’, ‘algorithms’, ‘propositions’, ‘diagrams’, etc., are being made by the historiographer, albeit in a reflectively controlled way, while at the same time those identifications presumably entrenched in the proceedings of the actors themselves under historical scrutiny are to be questioned and contextualized. If there is any chance then that a critical history of historiography of mathematics may be useful to historians of mathematics, it is only as a fine-grained approach to past historiographies down to the particulars of their infrastructure. In so doing, it should also qualify as an account of the ways in which distinctive historiographic strands may have interwoven, at that infrastructural level, in the shaping of provisionally cumulative sequences, through various kinds of distortions and cultural transfers.

The present paper aims at contributing to such critical and contextual history of historiography of mathematics. It analyzes the ways in which nineteenth-century European scholars attempted to make sense of Sanskrit mathematical sources dealing with what was acknowledged by the actors themselves, from the very outset, as ‘indeterminate analysis’. As for other subfields of mathematics, but, as will be seen below, perhaps even more acutely, the historiography of indeterminate analysis raised, from its very inception, a range of specific methodological problems which may be gathered under the heading of historiography of comparison. Issues pertaining to the perplexing balance between universalism and particularism in history writing, or the vexing trade-off between transhistorical mathematical content and culture-bound methods and practices, recurrently came to the fore, and were discussed in connection with the available evidence in significantly different intellectual, social and institutional contexts. By scrupulously reconstituting the sequence of receptions and innovations, we will attempt to unfold the historical processes by which various historiographical approaches to those Sanskrit sources were sifted through, resulting in a presumably cumulative, although revisable, historiography of the field.

In the course of the 1810s, three British scholars, all settled at that time in colonial India, and none a trained mathematician, Edward Strachey, John Taylor and Henry Thomas Colebrooke, published English translations of Sanskrit mathematical works by the twelfth-century Indian astronomer and mathematician Bhāskara and his seventh-century predecessor Brahmagupta. These translations and the essays and dissertations that came along were later read in European scholarly circles, where they attracted the attention

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6 Cf. Goldstein (1976, pp. 140–143). This passage and the distinction made there are highlighted by Aviezer Tucker in his philosophy of historiography, see Tucker (2004, pp. 6–7).
9 Cf. Goldstein (2010, pp. 5–6): “C’est sur le repérage de certaines identités que se fonde toujours la possibilité d’écrire l’histoire d’une découverte mathématique: écrire l’histoire de l’algèbre suppose que soit identifié ce qu’est l’algèbre, ou à tout le moins ce qui pourrait relever de cette histoire particulière”. The same holds for the history of any subfield of mathematics, as for instance indeterminate analysis. One should be able to account for the decision to use the same generic denomination for what is found in Sanskrit, Greek, Arabic and eventually European sources.
of both mathematicians and philologists throughout the nineteenth-century. Among the British Indologists, Strachey deserves special consideration for our current purpose, not only because he initiated the series of English translations of major Sanskrit mathematical treatises, but also and mainly because he first identified a consistent algebraic theory in Bhāskara’s works. While reading his sources through the conjectural prism of the Euler–Lagrange theory of periodic continued fraction expansions for quadratic irrationals, he offered an insightful interpretation of the paramount achievement of the Hindus in indeterminate analysis, the so-called cakravāla, or “cyclic method” – a method which the German mathematician and historian of mathematics, Hermann Hankel (1839–1873), would later hail as “the climax of all Hindu science”, a method which “is beyond all praise; [and] is certainly the finest thing achieved in the theory of numbers before Lagrange.” Two decades after Strachey, the French geometer Michel Chasles interpreted the Sanskrit methods in indeterminate analysis in a completely different way. While turning to Sanskrit sources in the context of his grand scale research program on the historiography of geometry, he was induced to put great emphasis on geometry. By shifting his main focus from Bhāskara to Brahmagupta, with whose works he acquainted himself through Henry Thomas Colebrooke’s translations from the Sanskrit original, which had become authoritative all over Europe in the meantime, he set out to restore what he took to be the true meaning of Brahmagupta’s geometry, viz. his theory of quadrilaterals. In the midst of this enterprise, a geometrical construction caught his eye, for it suggested a geometrical solution of the indeterminate equations of the second degree, which Chasles conjectured may have been the Sanskrit path to indeterminate analysis. Our focus in the following pages will be on both these interpretive conjectures which jointly shaped the way nineteenth-century historiography of indeterminate analysis gradually absorbed Sanskrit sources, from Strachey to Chasles, and from Chasles to Hankel, with a constellation of scholars, Charles Hutton, Olry Terquem, Henry Thomas Colebrooke, Guglielmo Libri, Baldassare Boncompagni, Franz Woepcke and others playing their parts in the background.

2. Edward Strachey and the Gem of Cakravāla

Edward Strachey (1774–1832) was an East India Company servant, who went to Bengal in 1793 and held there various diplomatic and judicial positions until his return to England in 1811. The second son of Sir Henry Strachey, first baronet of Sutton Court, he matriculated at St Andrews University in 1790–1791. In these years, Nicolas Vilant (1737–1807) was Regius professor of mathematics there, but, owing to his poor health, he delegated most of his teaching duties to successive assistants, among whom John West (1756–1807) stands out with hindsight as one of the few early exponents of continental analysis in Britain, long before the short-lived Cambridge Analytical Society was even constituted in 1812. However, the

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10 In the present paper, the international standard ISO 15919 for transliterating Sanskrit is adopted.
11 Cf. Hankel (1874, pp. 200–202); “der Glanzpunkt ihrer gesammten Wissenschaft . . . Diese Methode ist über alles Lob erhoben; sie ist sicherlich das Feinste, was in der Zahlentheorie vor Lagrange geleistet worden ist.”
12 Cf. Sanders (1968). The Strachey family counted many British civil servants, some soldiers posted in colonial India, and a series of baronets of Sutton Court, as well as a few scholars and literary figures: one of Edward Strachey’s ancestors was a friend of John Locke, his great-grand father a famous geologist, while his grandson would later be none other than Lytton Strachey (1880–1932), the well-known British writer and critic, member of the Bloomsbury group, and friend of John-Maynard Keynes, Bertrand Russell, and Virginia Woolf.
14 On the teachings of John West, see Craik (1998, pp. 30–31): “[West] acquired an intimate knowledge of works by Laplace, Lagrange, and Arbogast at a time when few in Britain had reached such a level.” He taught mathematics at St Andrews from 1775 to 1784, and James Brown from 1784 (presumably) to 1796, cf. Craik (2012, p. 176).
15 Let us recall that Charles Babbage and George Peacock were both born in 1791, that is at about the time Edward Strachey began his education at St Andrews University. On the Cambridge Analytical Society and the promotion of analytics in Britain in the 1810s and 1820s, see Enros (1981, 1983).
young Strachey studied at St Andrews at a time when West had already left Scotland for Jamaica, and had been replaced as Vilant’s assistant by James Brown, himself a former student and later a friend of West. While being possibly taught at least the essentials of continental analysis by Brown, Strachey may have become aware at that time already of the work of Lagrange, which was barely known then among British mathematicians.\(^{16}\) Whatever the case may be, during his stay in India, Strachey translated into English a Sanskrit treatise on algebra, the \(\text{Bija-gan\!\!\!i\!\!\!ita}\), by the twelfth-century Indian astronomer and mathematician Bhāskara. Knowing Persian,\(^{17}\) but unable to understand Sanskrit, he based his work on a Persian translation which was made in India in 1634, by an author referred to as “Ata Allah Rusheedee”. However, Strachey warned, “the Persian [\(\text{did}\) not in itself afford a correct idea of its original, \(\ldots\); for it is an undistinguished mixture of text and commentary, and in some places it even refers to Euclid. \(\ldots\) [\text{And yet}]\) a little patience will discover evidence of the algebra which it contains, being purely Hindoo”.\(^{18}\) So as to sort out the core text stemming from the original in Sanskrit from later interpolations, Strachey used sets of notes by two elder scholars who had access to Sanskrit manuscripts. Reuben Burrow (1747–1792) and Samuel Davis (1760–1819) were both close friends of William Jones (1746–1794),\(^{19}\) the founder the Asiatic Society of Bengal, and early contributors to the \(\text{Asiatick Researches}\), the journal Jones edited, “inquiring \(\text{[\text{as the full title went –]}\) into the history and antiquities, the arts and literature of Asia”. They assisted him in carrying out a research program aiming at settling Indian chronology on the basis of events presumably registered in ancient astronomical texts.\(^{20}\) While investigating these sources, they were struck by the high scientific attainments of the ancient Hindus. Burrow observed for instance that “the Brahmins calculate their eclipses, not by astronomical tables as we do, but by rules, \(\text{[which presumably yielded]}\) a proof that they must have carried algebraic computation to a very extraordinary pitch”,\(^{21}\) and hence suggested the existence of an

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\(^{16}\) Although, as will be seen below, Charles Hutton would later play an important role in spreading Strachey’s views on Hindu algebra, it is not clear whether Strachey had read Hutton at an earlier stage of his own work, and hence whether he may have been induced thereby to turn his attention to Euler and Lagrange, for Hutton was also an early admirer of the analytical methods, cf. Guicciardini (1989, p. 112): “Hutton’s Dictionary [\(\text{viz.} \text{(Hutton, 1795)}\)] reveals a deep and extensive knowledge of continental methods (\(\ldots\)) Hutton gave space to d’Alembert, Euler and Lagrange, providing an outline of their methods and results. However, Hutton’s Dictionary did not include technical details of the foreign works.”

\(^{17}\) In his \textit{Reminiscences}, the Scottish philosopher, satirical writer and historian, Thomas Carlyle (1795–1881) recounts a journey to Paris with Strachey, and in particular a visit to Antoine-Léonard de Chézy (1773–1832), who had learnt Arabic and Persian with Antoine-Isaac Silvestre de Sacy (1758–1838), and later held the chair of Sanskrit created at the College de France in 1815. Cf. Carlyle (1881, I, p. 271): “Strachey and I went one evening to call upon M. de Chézy, Professor of Persic, with whom he, or his brother and he, had communicated while in India. We found him high aloft, but in a clean snug apartment, burly, hearty, glad enough to see us, only that Strachey would speak no French, and introduced himself with some shrill sounding sentence, the first word of which was clearly \textit{salaam}. Chézy tried lamely for a pass or two what Persian he could muster, but hastened to get out of it, and to talk even to me, who owned to a little French, since Strachey would own to none.”

\(^{18}\) Cf. Strachey (1813, pp. 4–5).

\(^{19}\) On the scholarly pursuits of the Anglo-Welsh philologist, linguist and Indologist, William Jones, see Cannon (1990) and Franklin (2011). A man of outstanding linguistic capacities, he first formulated the program of Indo-European linguistics. On February 1786, only six months after his first steps in Sanskrit, in his ‘Third Anniversary Discourse’ read before the Asiatic Society, he suggested that “the Sanskrit language, whatever be its antiquity, is of a wonderful structure: more perfect than the Greek, more copious than the Latin, and more exquisitely refined than either, yet bearing to both of them a stronger affinity, both in the roots of verbs and in the forms of grammar, than could possibly have been produced by accident; so strong indeed that no philologer could examine them all three, without believing them to have sprung from some common source, which perhaps no longer exists.” (quoted by Franklin, 2011, p. 36). However, in contrast to later German comparative grammarians, Jones’s aim was to use linguistic evidence to support the monogenetic view of mankind descending from Noah’s sons, Shem, Ham and Japhet, as told in the Bible.

\(^{20}\) On William Jones’s work on the ancient scientific traditions of the Hindus, and in particular his interactions with Indian pandits, see Rocher (1995).

Indian algebraic tradition. Approximations used in astronomy presumably testified to an acquaintance with infinite series. Burrow later discovered that the Hindus possessed a more accurate method of calculating the parallaxes of the moon than that provided by modern nautical almanacs, which led him to claim that they surpassed the Westerners in significant tracts of algebra and already knew the binomial theorem. The computational procedures used by the Hindus in astronomy also attracted considerable attention on the part of Davis in these years. In this connection, Jones urged him to learn enough Sanskrit so as “not to be at the mercy of pandits” or at least so as to “control the process as firmly as he did”, so that he might eventually provide more accurate astronomical tables than his predecessors Guillaume Le Gentil, or Jean-Sylvain Bailly. Dhruv Raina points out that, in contrast to the French savants, the British Indologists “engaged with specific texts and from the astronomical rules presented there made a claim that these rules must be based on a mathematical system, and proceeded to discover mathematical texts – [thus shifting] their focus from the origins of astronomy to the origins of Indian mathematics, and in particular Indian algebra and arithmetic.”

On May 4, 1789, William Jones wrote to Davis to stress how much was to be expected from a thorough knowledge of Bhāskara’s Bīja-ganita for the early history of Indian algebra:

I anxiously hope that the work of Bhāskara may prove a treatise on universal arithmetic. … I have met, in the Lettres Édifiantes, a curious passage on Indian science, which you will soon be able to disprove or confirm: ‘The Hindu logicians’, says Father Du Pons, ‘admit of four principles of knowledge; 1. prátyacsha or intuition. 2. infallible or divine authority. 3. anumána, which means syllogism or enthymema. 4. upamána, or equation, which is the application of a definite known quantity to the definition of another quantity till then unknown.’ Now a clearer description of algebra than this could hardly be given; if there be treatises on specious arithmetic in Sanskrit, we shall possibly find rules and methods, which may be substantially useful.

In this context, both Burrow and Davis studied Bhāskara’s Bīja-ganita, and yielded valuable interpretive material upon which Strachey drew with the purpose of “support[ing] the opinion that the Hindoos had an original fund of Science not borrowed from foreign sources”. Regarding the rule for constructing the sines by differences which they employed in astronomy, Strachey for instance strongly denied that the Hindus were “totally ignorant of the principles of the operation”, as the Scottish mathematician John Leslie

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22 Cf. Burrow (1788).
23 Cf. Burrow (1790).
24 Cf. Davis (1790).
27 Cf. Jones and Davis (1831, p. 8). Jones refers to Jean-François Pons (1698–1753), a French Jesuit missionary in India and also a Sanskritist, who wrote on Indian science in 1743. However, only Jones discerned a characterization of algebra in Pons’ corresponding account, see Pons (1743, pp. 224–243): “ils [the ancient Hindous] admettaient, comme les Modernes, quatre principes de Science: le témoignage des sens bien expliqués, Pratyaksham; les signes naturels, comme la fumée l’est du feu, Anumānam; l’application d’une définition connue au défini jusques-là inconnu, upamānam; enfin, l’autorité d’une parole infaillible aptachabdham.”
28 Cf. Strachey (1813, p. 1).
29 Cf. Leslie (1809, p. 485).
30 John Leslie (1766–1832) succeeded John Playfair (1748–1819) as professor of mathematics at Edinburgh University in 1805. In this transitional period for Scottish mathematics, as Euclidean geometry and algebraic analysis afforded conflicting mathematical ideologies, his preference went to geometrical analysis which he deemed logically sounder. A revealing controversy opposing John Leslie to the analysts William Wallace and James Ivory, broke out in the early 1820s in the aftermath of the publication of Thomas Carlyle’s English translation of Legendre’s Éléments de géométrie, cf. Craik (2000). Thomas Carlyle, who had once been a student of Leslie’s and first envisioned a career as a mathematician before turning to history, failed to get grasp on continental calculus.
had contended, and hence that they were merely “humble calculators ... content to follow blindly a slavish routine.”31 While observing that most of the theorems on which the operations of the Hindus depended are given in the form of rules, he claimed that “it is not to be inferred because the demonstrations do not always accompany the rules, therefore that they were not known; [for] on the contrary, the presumption in such a case is that they were known.”32 His main argument in favor of the authenticity of a presumably “Hindoo algebra in a Persian dress”33 pressed the point of consistency. Not only did old mathematical Sanskrit manuscripts happen to be “exceedingly scarce” at the time, but the most recent ones also often contained interpolations from Greek, Arabic, and modern European knowledge, so that sorting out what is of Indian origin presumably required “recur[ring] to the nature of the propositions themselves.”34 One would thus arguably find in Sanskrit sources “a perfectly connected structure of science, comprehending propositions which in Europe were invented successively by Bachet de Mezeriac [sic], Fermat, Euler, and De La Grange.”35 Strachey hence spelled out four propositions of indeterminate analysis forming the core of what the ancient Hindus supposedly achieved on their own in algebra:

1. A general method of solving the problem \( \frac{ax^2 + bx + c}{b} = y \), \( a \), \( b \) and \( c \) being given numbers and \( x \) and \( y \) indeterminate. The solution is founded on a division like that which is made for finding the greatest common measure of two numbers. The rules comprehend every sort of case, and are in all respects quite perfect.
2. The problem \( am^2 + 1 = n^2 \), \( a \) being given and \( m \) and \( n \) required with its solution.
3. The application of the above to find any number of values of \( ax^2 + b = y^2 \) from one known case.
4. To find values of \( x \) and \( y \) in \( ax^2 + b = y^2 \) by an application of the problem \( \frac{ax^2 + bx + c}{b} = y \).36

Interestingly enough, Strachey adduced evidence “that it is not all forgery” from Lagrange, who not only gave Bachet credit for first addressing the problem of solving equations in two or more unknowns in whole numbers, but also outlined the subsequent unfolding of what he took to be an emerging topic up to his own results of 1768. Strachey essentially relied on Lagrange’s historical account37 which he extensively quoted in Edinburgh at that time, and his frustration bears witness to the social context of Scottish mathematics in these years. Leslie criticized Legendre’s analytical approach to geometry which he considered at odds with his own, resolutely modeled upon ancient Greek geometry, as first propounded in his Elements of Geometry, Geometrical Analysis, and Trigonometry (1809), cf. Leslie (1809, p. viii–ix): “The analytical investigations of the Greek geometers are indeed models of simplicity, clearness, and unrivaled elegance; and though miserably defaced by the riot of time and barbarism, they will yet be regarded ... as some of the noblest monuments of human genius. It is a matter of deep regret, that Algebra, or the Modern analysis, from the mechanical facility of its operations, has contributed, especially on the Continent, to vitiate the taste and destroy the proper relish for the strictness and purity so conspicuous in the ancient method of demonstration.” Leslie also authored The Philosophy of Arithmetick (1817), viz. (Leslie, 1817), upon which Alexander von Humboldt elaborated in his work on the origin of the positional notation system, viz. (von Humboldt, 1829).

31 Cf. Leslie (1809, p. 425). The Indians could not have borrowed this supposed “slavish routine for the sines” from their neighbors, Strachey responded, for it depended on “a principle not known even to the modern Europeans till 200 years ago”, cf. Strachey (1813, p. 2). As a rule, not only did Leslie highly praise the Greeks, he also “attacked the Hindoos”, something Strachey deplored. In the last part of his 1809 Elements of Geometry, Leslie exposed the geometrical principles underlying the construction of trigonometrical tables. After explaining the rule for constructing the sines by differences, which Davis had presented as the method of the ancient Hindus, (Davis, 1790, pp. 245–248), he sounded the above deprecative note so as to downplay their achievement, cf. Leslie (1809, pp. 485–6).
32 Cf. Strachey (1813, p. 2).
33 Cf. Strachey (1813, p. 8).
34 Ibid.
35 Cf. Strachey (1813, p. 6).
36 Cf. Strachey (1813, p. 6).
37 Cf. Lagrange (1770b, pp. 655–658): “La plupart des Géomètres qui ont cultivé l’Analyse de Diophante se sont, à l’exemple de cet illustre inventeur, uniquement appliqués à éviter les valeurs irrationnelles; & tout l’artifice de leurs méthodes se réduit à faire en sorte que les grandeurs inconnues puissent se déterminer par des nombres commensurables. L’art de résoudre ces sortes
and appropriated for his own purpose\textsuperscript{38} by making perspicuous the parallelism with the theory presumably contained in Bhāskara’s \textit{Bīja-ganita}. After Bachet solved what corresponds to the first problem of the Hindus, Fermat challenged the English mathematicians with what supposedly amounted to the second problem listed above. Although they solved Fermat’s special problem, yet the English did not grasp its significance for the general solution of indeterminate problems of the second degree. Only Euler saw how it provided the key for finding an infinite number of solutions to the general quadratic equation in two unknowns out of a previously known solution, that is the third problem presumably solved by the Hindus. Eventually, Lagrange himself yielded a new method for solving the indeterminate problems of the second degree by means of periodic continued fraction expansions. As Lagrange’s mathematical theory enabled him to discern a consistent body of knowledge in his sources, Strachey also settled the question of the chronology by calling upon Davis’s authentication of his notes as based on the original Sanskrit.\textsuperscript{39} “It is true that Bachet wrote a few years before 1634 [\textit{viz. the inferred date of the Persian translation Strachey worked on,}] but it is no sort of objection to the argument. . . . Mr Davis’s notes shew that it is in the Sanskrit Bija Ganita, which was written four centuries before Bachet.”\textsuperscript{40}

Strachey’s insightful mathematical reading of Bhāskara’s \textit{Bīja-ganita} from a seventeenth-century, rather distorted, Persian version is a feat which may best be appreciated by comparison with Davis’s scarce notes\textsuperscript{41} and Colebrooke’s complete translation from the Sanskrit to be issued a few years later. According to Strachey, the indeterminate analysis of the Hindus would culminate in the so-called “cyclic method”, or \textit{cakravāla}, which, with hindsight, may be seen as aptly articulating two distinct ingredients: 1. the so-
called bhāvanā rules, or rules of production or composition, on the one hand, and 2. the kuṭṭaka, or the “pulverizer”, that is the method for solving in integers first-degree problems of the type \( \frac{ax+c}{b} = y \), \( a, b \) and \( c \) being given whole numbers [viz. prop. I above]. Strachey successfully grasped general procedures from the rules being merely stated and the examples worked out, as a result of his identifying in the Persian text, the key technical Sanskrit words which Davis had stressed and for which he had provided English equivalents. In the absence of a symbolic language, stereotypic terminology proves of paramount importance as a way to carry the generality of the procedures. Problems connected to equations of the type \( Na^2 + k = b^2 \) were referred to as vargaprakṛti (“square-natured”), \( N \) being the prakṛti (the “nature”); \( k \), the kṣepa (the “additive”, or more generally the “interpolator”, which can be positive or negative); \( a \), the kaniṣṭha (the “least root”); and \( b \) the jyesṭha (the “greatest root”). Rules would then be stated and procedures carried out by means of these terms referring to specific quantities with which one would proceed according to certain stipulations. The bhāvanā rules, for instance, were thus spelled out as computational procedures operating on quantities written in successive rows, and yielding results also presented in a further row.

<table>
<thead>
<tr>
<th>Least Root</th>
<th>Greatest Root</th>
<th>Interpolator</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>( b )</td>
<td>( k )</td>
</tr>
<tr>
<td>( c )</td>
<td>( d )</td>
<td>( k' )</td>
</tr>
<tr>
<td>( ad + bc )</td>
<td>( bd + Nac )</td>
<td>( kk' )</td>
</tr>
</tbody>
</table>

The first two rows correspond to the equations \( Na^2 + k = b^2 \) and \( Nc^2 + k' = d^2 \), respectively; and the rule describes the procedure to form a fresh pair of “least” and “greatest” roots for the same “nature” \( N \), together with the appropriate “interpolator”, corresponding to the equation

\[
N(ad + bc)^2 + kk' = (bd + Nac)^2.
\]

This is the so-called “positive” bhāvanā rule, whereas by contrast the “negative” one yields \( ad - bc \) as the “least” root, and \( bd - Nac \) as the “greatest” root. Both bhāvanā rules then correspond to the following identities, sometimes referred to as Brahmagupta’s identities:

\[
(b^2 - Na^2)(d^2 - Nc^2) = (bd \pm Nac)^2 - N(ad \pm bc)^2.
\]

Since one can operate with the same pair twice, or with two different pairs of roots, it is clear that repeated application of the bhāvanā rules allows one to generate any number of solutions to the equation

---

42 In Strachey’s translation of the Persian treatise, problems of indeterminate analysis of the first degree are dealt with in chapter VI. The bhāvanā rules which belong there occur in the opening pages of this chapter, cf. Strachey (1813, pp. 36–37). In Colebrooke’s translation from the Sanskrit, they are exposed in the first section of chapter III, “Affected square” [vargapraṇāti], cf. Colebrooke (1817, §§77 and 78, p. 171).

43 As for Strachey’s translation from the Persian, the kuṭṭaka method is presented in chapter V; see Strachey (1813, pp. 29–30) for the rules, and Strachey (1813, pp. 30–36) for the examples. The name “pulverizer” is due to Colebrooke, and the method is detailed in chapter II of his translation of the Bija-ganita, cf. Colebrooke (1817, art. 54–57, pp. 156–159). For a modern presentation, see Datta and Singh (1962, II, pp. 113–114).


45 About these equations, André Weil suggests that “for us, perhaps the easiest way of verifying [hem]” is to write them as a product of four quadratic binomial irrationalities, viz. \((x + \sqrt{N}y)(x - \sqrt{N}y)(z + \sqrt{N}t)(z - \sqrt{N}t)\), to be multiplied pairwise in two different ways, as “was first pointed out by Euler in his Algebra of 1770 (Eu. I. 1. 422, Art. 175)”, cf. Weil (1984, pp. 14–5). However, in his translation, Colebrooke provides a proof by Krṣaṇa, a seventeenth-century commentator of Bhāskara, which proceeds by merely multiplying both sides of the equations of the vargaprakṛti type by suitable squares and making appropriate substitutions, cf. Colebrooke (1817, p. 172).
\(N x^2 + k = y^2\), out of a previously known pair of roots \(Na^2 + k = b^2\), and any known solution of the equation of the same “nature” with the unit as “interpolator”, that is the equation usually referred to as Pell’s equation, \(N x^2 + 1 = y^2\). By contrast, the bhāvanā rules do not in general suffice (versely) to derive a solution of Pell’s equation \(N x^2 + 1 = y^2\), from a previously known solution of the corresponding equation with the “interpolator” \(k\), viz. \(Na^2 + k = b^2\). A new method is required for this, the so-called “cyclic method”, or cakravāla,\(^{47}\) from cakra, a wheel or a circle, a method which may be schematized in the following way.

Let a given “nature” \(N\) be fixed. Assuming a pair of roots \((a, b)\) for the “interpolator” \(k\), viz. \(Na^2 + k = b^2\), where \(a, b, k\) are relatively prime integers, \(a\) and \(b\) being positive, and \(k\) positive or negative, the step rule yields another pair of roots \((a_1, b_1)\) for a new “interpolator” \(k_1\), viz. \(Na_1^2 + k_1 = b_1^2\). The whole method then consists in iterating the procedure until a pair of roots \((a_n, b_n)\) is obtained for the “interpolator” \(k_n = 1\), viz. \(Na_n^2 + 1 = b_n^2\).\(^{48}\) As for the step rule itself, one proceeds as follows. Starting from a given solution \(Na^2 + k = b^2\), one can always find another solution for the same “nature” \(N\), for in any case \(N I^2 + (m^2 - N) = m^2\). As a matter of fact, one picks up the closest square \(m^2\) to \(N\) satisfying a further condition to be stipulated below. One then applies the bhāvanā rules to both these solutions.

<table>
<thead>
<tr>
<th>Least Root (a)</th>
<th>Greatest Root (b)</th>
<th>Interpolator (k)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(am + b)</td>
<td>(bm + Na)</td>
<td>(k(m^2 - N))</td>
</tr>
</tbody>
</table>

The last row corresponds to the equation \(N(am + b)^2 + k(m^2 - N) = (bm + Na)^2\), from which immediately follows that

\[
N \left( \frac{am + b}{k} \right)^2 + \frac{m^2 - N}{k} = \left( \frac{bm + Na}{k} \right)^2
\]

\(^{46}\) In the following cases \(k = -1, k = \pm 2\) and \(k = \pm 4\) though, Brahmagupta had previously devised a method to derive a solution to Pell’s equation \(Na^2 + 1 = b^2\) from any solution for the “interpolator” \(k\), viz. \(Na^2 + k = b^2\), by merely iterating the bhāvanā rules, as Colebrooke’s translation of Brahmagupta later revealed, cf. Colebrooke (1817, pp. 365–366), chap. XVIII, sec. VII, Art. 69, Rule §42, and Art. 71, Rule §43. The idea of the method is simple; if one can get a perfect square as an interpolator, then one can divide all three values, both roots and the interpolator by this perfect square, hence obtain a solution with the same “nature” \(N\), but this time with interpolator 1. Colebrooke’s translation of Bhāskara’s Bhīja-ganita registers these methods, but neither Strachey, nor Davis apparently ever mention them.

\(^{47}\) This method is presented in Bhāskara’s Bhīja-ganita, but it is also to be found in the earlier works of Jayadeva, an eleventh-century author. “Its true originator remains unknown”, cf. Weil (1984, p. 22). Strachey would call it “the operation of circulation”, cf. Strachey (1813, pp. 41–44), and Davis devoted one of his notes to the so-called “chakra balā”, cf. Strachey (1813, pp. 102–103). In Colebrooke’s translation, the “cyclic method” is the topic of the second section of chapter III, devoted to vargaprakṛti, cf. Colebrooke (1817, pp. 175–178). The rule is presented in Art 85–86, and examples follow. Colebrooke inserts in a footnote the following commentary by Śūryadāsa, a sixteenth-century commentator of Bhāskara’s Bhīja-ganita: “Chacravāla, a circle; especially the horizon. The method is so denominated because it proceeds as in a circle: finding from the roots (“greatest” and “least”) a multiplier and a quotient (by Chapter 2 [viz. by the kuṭṭaka method]); and thence new roots; whence again a multiplier and a quotient, and roots from them; and so in a continued round.”

\(^{48}\) Still, Bhāskara’s Sanskrit text, as translated by Colebrooke, stipulates that the process of cakravāla is to be iterated until one hits upon any “interpolator” of the form \(k = \pm 1, \pm 2, \pm 4\), to which one can then apply the bhāvanā rules, as is the case with Brahmagupta’s method for instance. However, as will be seen below, the translation from the Persian source does not mention this shortened normalized version of the cakravāla method. “Actually this [viz. Bhāskara’s halting clause \(k = \pm 1, \pm 2, \pm 4\)] is no more than a shortcut, [André Weil emphasizes,] since it can be shown that the cakravāla, applied in a straightforward manner, would inevitably lead to a triple \((p, q, 1)\) as desired [viz. \(Np^2 + 1 = q^2\)]; while this shortcut is quite effective from the point of view of the numerical solution, it destroys the “cyclic” character of the method, which otherwise would appear from the fact that the “additives” \(m, m', m''\) . . . would repeat themselves periodically, corresponding to the periodicity of the continued fraction of \(\sqrt{N}\).” Cf. Weil (1984, p. 24).
By means of the *kutṭaka* method, one then determines the values of the integer \( m \) for which \( \frac{am+b}{k} \) is also an integer, so that one may eventually choose among these values that one which makes \( |m^2 - N| \) as small as possible. One thus obtains three new integers\(^{49}\)

\[
a_1 = \frac{am+b}{k}, \quad b_1 = \frac{bm+Na}{k}, \quad k_1 = \frac{m^2 - N}{k}
\]

from which the whole procedure can be started again, until a certain halting condition is met. However, Strachey’s and Colebrooke’s versions from the Persian and the Sanskrit respectively, differ significantly on this score. Whereas Colebrooke has it explicitly that the iterative procedure stops with any “interpolator” \( k = -1, k = \pm 2 \) and \( k = \pm 4 \), for then the *bhāvana* rules would take over and bring the process to its end with \( k = 1 \); Strachey apparently proves at a loss when it comes to interpreting in a mathematically consistent way the Persian phrase for the halting condition, viz. in his translation: “*work as before till the original augment or the augment of the square is found*”.\(^{50}\) In spite of this difficulty, Strachey perceived the analogy between the *cakravāla* and the modern algorithm of continued fractions, owing essentially to his relying on the few examples being fully spelled out in his source. Davis already remarked on this score: “I find no abstract of the rule for the ‘operation of circulation,’ but there is the first example, viz. 67\(x^2 + 1 = \Box\).”\(^{51}\)

Indeed, in the case of \( N = 67 \), the *cakravāla* method starts with 67.1\(^2 - 3 = 8^2 \), that is with the “least” value 1, “greatest” value 8, and “interpolator” –3, viz. the triple (1, 8, –3), and successively generates the triples (5, 41, 6), (11, 90, –7), (27, 221, –2), and eventually (5967, 48842, 1), which provides the solution to 67\(x^2 + 1 = y^2 \). By comparison, Euler’s algorithm yields the periodic continued fraction\(^{53}\)

\[^{49}\text{Although neither Strachey’s nor Colebrooke’s accounts seem to provide any hint of a proof for this, divisibility considerations make it clear that from } Na^2 + k = b^2 \text{ and } \frac{am+b}{k} \text{ being an integer, there follows that } a_1, b_1 \text{ and } k_1 \text{ are all integers. As will be seen below, Hankel first proved this fact owing to the symmetry of the positive and the negative } bhāvana \text{ rules, thereby providing a proof of correctness for the iterated step of the } cakravāla \text{ procedure.}\]

\[^{50}\text{Cf. Strachey (1813, p. 43). The method as presented in Strachey’s source dealt with a more general problem than Pell’s equation, viz. solving in integers the equation } Ax^2 + B = y^2 \text{ with “interpolator” } B \text{ [cf. prop. 4 above]. The “cyclic method” would start with any triple } (f, \beta, g), \text{ solution to } A\beta^2 + \beta = g^2, \text{ and would generate a series of triples } (f', \beta', g'), (f'', \beta'', g''), etc. \text{ Then Strachey reads the halting condition as meaning: “If } \beta' \text{ is neither } = B \text{ nor to } BP^2, \text{ nor to } P^2_b \text{ proceed as before.”, cf. Strachey (1813, p. 42). And the above formulation of the halting condition prompts the following comment on the part of Strachey: “I suppose it should be } BP^2 \text{ or } P^2 \beta. \text{ I think it likely that this does not form a part of the original rule which seems to relate to integer values only”, cf. Strachey (1813, p. 43). Since, indeed, the gist of the whole method is to solve the equation in integers, the halting condition remained shrouded with uncertainty in Strachey’s version.}\]

\[^{51}\text{Cf. Strachey (1813, p. 103).}\]

\[^{52}\text{Strachey’s and Colebrooke’s translations fully agree as regards all the stages of the method applied in the case } N = 67, \text{ cf. Strachey (1813, pp. 43–44), Colebrooke (1818, §87, pp. 176–178).}\]

\[^{53}\text{In a paper of 1759, published in 1767, Euler devised a new algorithm for generating the continued fraction expansion of quadratic surds } \sqrt{N}, \text{ applied it to many values of } N \text{ and tabulated the results thus computed. In the case of } N = 67, \text{ Euler proceeded in this way, cf. Euler (1759) 1767, pp. 81–82): one first observes that } 8 \text{ is the greatest integer less than } \sqrt{67}, \text{ viz. } \sqrt{67} = 8 + \frac{1}{a}; \text{ hence } a = \frac{1}{\sqrt{67} - 8} = \sqrt{\frac{67}{6} + \frac{8}{6}} = 5 + \frac{1}{5}; \text{ then } b = \frac{3}{\sqrt{67} - 7} = \sqrt{\frac{67}{6} + \frac{7}{6}} = 2 + \frac{1}{2}, \text{ etc. Euler remarked that for such quadratic surds, the continued fraction expansion is periodic and palindromic } \sqrt{N} = [a_0, a_1, a_2, \ldots, a_2, a_1, 2a_0]. \text{ He explored the properties of their finite convergents } [n_0, n_1, n_2, \ldots, n_k] = n_0 + \frac{1}{n_1 + \frac{1}{n_2 + \ldots + \frac{1}{n_k}}}. \text{ He for instance noticed that the polynomials corresponding to the penultimate convergent of the first period, viz. } \frac{P_k}{Q_k} = [a_0, a_1, a_2, \ldots, a_2, a_1] \text{ satisfy the equation } P_k^2 - N Q_k^2 = (-1)^{k+1}, \text{ while those corresponding to the penultimate convergent of the second period satisfy } P_{2k+1}^2 - N Q_{2k+1}^2 = (-1)^{2k+2} = 1, \text{ hence that every solution of Pell’s equation arises as the penultimate convergent of an even number of periods in the continued fraction. For a presentation of the basic theory of continued fractions in a historical perspective, see Fowler (1987, chap. 9). “While Euler drew attention to these properties of the continued fractions for the square roots } \sqrt{N}, \text{ as well as to their use in solving Pell’s}\]

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\[ \sqrt{67} = [8, 5, 2, 1, 1, 7, 1, 1, 2, 5, 16] \]

whose successive convergents may be seen as encapsulating the stages of the cakravāla, for each convergent of the continued fraction corresponds to an equation of the form \( N \alpha^2 + k = \beta^2 \). In the case \( N = 67 \) for instance, Euler’s algorithm successively yields ten equations ranging from \( 67.1^2 - 3 = 8^2 \) to \( 67.5967^2 + 1 = 48842^2 \), five of which correspond to the iterated steps of the cakravāla, both methods having the same starting and end points. However, in modern times, only Lagrange organized these facts into a consistent mathematical theory and in particular proved that quadratic irrationals have periodic continued fraction expansions, therefore providing Strachey with a neat makeshift solution to the halting problem for the cakravāla, albeit a rather anachronistic one. “This rule, [Strachey averred,] though in some respect imperfect [as regards, for instance, the halting problem], is in principle the same as that for solving the problem in integers by the application of continued fractions, which was first given by De La Grange.” However, Strachey apparently distinguished the underlying principle and the concrete layout of the method, as is revealed by the following observation: “As for the 4th of the points abovementioned, the method in detail (however imperfect in some respects) is, as far as I know, new to this day.” A few years later, he struck again such a cautious note, when noting that inquiries of this kind “might, perhaps shew that the Indians had a knowledge of continued fractions … for the foundation of the indeterminate analysis of the Hindus is directly explicable on the principle of continued fractions.” However, Strachey’s tentative and balanced account gave way on the part of his followers to an increasingly straightforward identification of the cakravāla method with Euler’s continued fraction algorithm, with or without some equivalent of Lagrange’s theory justifying the whole procedure – an identification which in return remained unquestioned until the twentieth century. And even then, mainstream historiography rather inclined to-

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55 Cf. Lagrange (1770a, §43, p. 615): “On avait remarqué depuis longtemps que toute fraction continue périodique pouvait toujours se ramener à une équation du second degré, mais personne que je sache n’avait encore démontré l’inverse de cette proposition; savoir, que toute racine d’une équation du second degré se réduit toujours nécessairement en une fraction continue périodique. Il est vrai que M. Euler, … a observé que la racine carrée d’un nombre entier se réduisait toujours en une fraction continue périodique; mais ce théorème, qui n’est qu’un cas particulier du nôtre, n’a pas été démontré par Euler, et ne peut l’être, ce me semble, que par le moyen des principes que nous avons établis plus haut.” Indeed, Lagrange’s proof first required establishing certain general formulas for the polynomials forming the convergents of the continued fraction.

56 Cf. Strachey (1813, p. 42).

57 Cf. Strachey (1813, p. 6).

58 Cf. Strachey (1818, p. 161).

59 One may mention the pioneering work of Krishnaswami Ayyangar (1892–1953), an Indian mathematician at the University of Mysore, who first attempted to assess the cakravāla as an original method in its own right, independently of the classical Euler–Lagrange method involving the regular continued fraction expansion of \( \sqrt{N} \). In a series of papers written between 1929 and 1941, Ayyangar attempted to frame a new type of continued fraction process which would “imitate” the cakravāla and open the way to an appreciation of the distance separating it from the Euler–Lagrange algorithm, see Ayyangar (1929–30), and D.H. Lehmer’s MR review of (Ayyangar, 1941). Clas-Olof Selénlius later continued and improved Ayyangar’s line of work, while trying to avoid some of his predecessor’s shortcomings, cf. Selénlius (1975). He devised a special kind of semiregular continued fraction process, which he called the “ideal” type, and which, unlike Ayyangar’s previous process, (1) applies generally to all real numbers.
ward doubting that there might have been ‘proofs’ in Sanskrit sources, hence also in the particular case of the cakravāla.  

In 1816, John Taylor, a Scottish missionary in Gujarat, then a government surgeon in Bombay, translated Bhāskara’s Līlāvatī, from the original Sanskrit, on the basis of three different manuscripts, the main one dating back to 1673 (the so-called Gujarat manuscript). Although he knew the previous work of Burrow and Strachey, “neither Strachey nor Colebrooke, first in Calcutta, later in London, were aware of Taylor’s research in Bombay.” As regards indeterminate analysis, Taylor paid attention to the two books on the kuttaka method, which “being seldom studied, are peculiarly dark and doubtful, [and thus make it] necessary, in most of the cases to go over the examples, as they are exhibited in the text and in the commentaries”. Although he refused to transpose Sanskrit mathematics into Western mathematical language, so as to avoid misinterpretations, he did not refrain from comparing the kuttaka with Euler’s method.  

Significantly enough, Taylor distanced himself from “an opinion not uncommon amongst the learned in Europe”, the view that the Hindus would have been unable to understand the rules they applied in their calculations. His conviction in this regard was shaped by his acquaintance with “the Udaharna [viz. the “Book of examples”], a work in [his] possession, contain[ing] demonstrations of many of the rules in the Lilawati.”  

In contrast to the contributions of such British Indologists as Burrow, Davis, Taylor, and even Strachey, the work of Henry Thomas Colebrooke (1765–1837) marked the real turning point in the making of Western Indology. His translations of Bhāskara and Brahmagupta from Sanskrit manuscripts published in 1817 and not only to numbers of the form \(\sqrt{D}\), and (2) presumably provides the rationale for the cakravāla process, “especially its ingenious core: the interaction between varga-prakṛti and kuttaka”, cf. Selenius (1975, p. 172–3). In this way, Selenius endorsed and deepened Strachey’s and Hankel’s insights regarding the cakravāla. “The old Indian cakravāla method [Selenius claimed,] . . . was a very natural, effective, labour-saving method with deep-seated mathematical properties. . . . More than ever are the words of Hankel valid, that the chakravāla method was the absolute climax (‘ohne Zweifel der Glanzpunkt’) of old Indian science, and so of all Oriental mathematics. . . . no European performances in the whole field of algebra at a time much later than Bhāskara’s, nay nearly up to our times, equalled the marvellous complexity and ingenuity of chakravāla”, cf. Selenius (1975, p. 180).

It is interesting to note in this respect that neither Ayyangar’s, nor Selenius’s work on the cakravāla were even mentioned by André Weil. Although he spent some time in India and developed connections with Indian mathematicians there in the early 1930s, cf. Weil (1991, chap. IV. India), there is no trace in his whole book on the history of number theory, of Ayyangar’s reassessment of the Indian method. On the one hand, Weil acknowledged the originality and the priority of Indian solutions to Pell’s equation, cf. Weil (1984, pp. 81–82): “What would have been Fermat’s astonishment if some missionary just back from India had told him that his problem had been successfully tackled there by the native Indians almost six centuries earlier?” But on the other hand, he also clearly denied that the Indian mathematicians may have gone beyond mere experimental knowledge on this score, cf. Weil (1984, p. 24): “For the Indians, of course, the effectiveness of the cakravāla could be no more than an experimental fact, based on their treatment of a great many specific cases, some of them of considerable complexity and involving (to their delight no doubt) quite large numbers. . . . Fermat was the first to perceive the need for a general proof, and Lagrange the first to publish one. Nevertheless to have developed the cakravāla and to have applied it successfully to such difficult numerical cases as \(N = 61\) or \(N = 67\) had been no mean achievement.”

Cf. Rocher and Rocher (2012, p. 133): “The fact that Colebrooke, Strachey and Taylor were concurrently engaged in translating the Līlāvatī and the Bijagranita – Strachey from a Persian version – speaks to the popularity of these texts, but also shows that there was little communication between these translators.”

Cf. Taylor (1816, p. 5).

Cf. Taylor (1816, p. 130): “The whole of kutaka corresponds with our method for the solution of indeterminate problems of the first degree, the same rules are given in Strachey’s Bijagranita, pages 29 and following. The process will also be found in Euler’s Algebra, vol. 2, p. 17 &c. with which Bhaskara’s may be compared.”

Cf. Taylor (1816, p. 36).

Cf. Taylor (1816, p. 37): “The information contained in this work corroborate the verbal information I have received, that the demonstrations, both in arithmetic and geometry, are performed by means of algebra; and that the Hindus never appear to have known or practiced the Grecian mode of analysis.”

set up the standards of philological rigor for further research in the field. They were read and studied by scholars all over Europe and soon became the reference on ancient Sanskrit mathematics. In the Dissertation prior to his translations, Colebrooke provided an overall account of the mathematical achievements of the ancient Hindus. There, he pointed to the solution of indeterminate problems of the second degree as one of the three main instances to be highlighted as the most significant ones, together with Pythagoras’ theorem and indeterminate problems of the first degree. On this score however, Colebrooke essentially endorsed Strachey’s conclusions, without offering a detailed comment or providing anything close to his predecessor’s thorough analysis of the Hindu methods. Still, his contribution also proved decisive in this respect, for he strengthened Strachey’s interpretation by substantiating it with his accurate translation from the Sanskrit and a wealth of illuminating notes drawing on the many commentaries he could use to supplement the main text. Furthermore, while dating Āryabhaṭa to the fifth century CE, prior to Brahmagupta, Colebrooke suspected that the most ancient Hindu algebraist then known “most probably” contributed to the indeterminate analysis of the second degree, in addition to the kuṭṭaka method. This mere presumption which on occasion turned into a claim in Colebrooke’s own writing, would later be challenged.

3. From India to Europe: early British and French reception

Spreading from India to Europe, the work of the British Indologists on Indian algebra aroused an interest in mathematical circles, even before Colebrooke’s publication, first in England then in France. This early reception was jointly shaped by two scholars from both sides of the Channel: Charles Hutton (1737–1823), a retired professor of mathematics at the Royal Military Academy at Woolwich, near London, who wrote mathematical textbooks and edited almanacs; and Oly Terquem (1782–1862), soon a pivotal figure in the milieu of the French écoles d’artillerie and later the editor with Camille Christophe Gerono of the Nouvelles annales de mathématiques, which, from 1855 to 1862, would be supplemented with a Bulletin de bibliographie, d’histoire et de biographie mathématiques promoting history of mathematics. Already well-known for his Mathematical and Philosophical Dictionary (1795), Hutton devoted one of his Tracts on mathematical and philosophical subjects (1812) to the history of algebra, a survey comprising a whole section on Indian algebra in which he analyzed “many valuable contributions ... by our own learned countrymen, belonging to the Bengal Society, ... as William Jones, Samuel Davis, Esq., Edward Strachey, Esq. and many others.”

Hutton was in direct contact with Davis and Charles Wilkins, the oriental librarian to the East India Company. Through Isaac Dalby, professor of mathematics at the Royal Military College at Wycombe, Hutton had also access to Burrow’s notes on Sanskrit mathematical sources, as well as to part of Strachey’s work on the Bija-ganita, more than a year before the latter’s book was issued in London in 1813. For his part, Terquem translated into French the section on Indian algebra of Hutton’s account, thereby making these researches known for the first time in France, a momentous initiative as Chasles later emphasized. “One of

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67 Colebrooke’s reception in German mathematical and philological circles has been dealt with elsewhere, cf. Smadja (2015).
68 See for instance Colebrooke (1817, pp. xiv–xv; xviii–xix).
69 Cf. Colebrooke (1817, p. vii; x).
70 Cf. Colebrooke (1817, p. xxi): “The Hindus in the fifth century, perhaps earlier, were in possession of Algebra extending to the general solution of both determinate and indeterminate problems of the 1st and 2nd degrees.”
72 Cf. Hutton (1812, p. 152).
73 Hutton used what he referred to as “the printed notices of Mr Strachey”, that is a short tract, from which he extensively quoted, titled Observations on the originality, extent, and importance of the mathematical science of the Hindoos, published at Calcutta in 1808. Besides, Hutton added, “Mr. Dalby has also lately received from Mr. Strachey, now in India, an entire English translation of the Bija, of a part of which he has favoured me with the perusal, and besides communicated by letter many descriptive remarks of those works, from all which sources I have collected the following curious particulars.” Cf. Hutton (1812, p. 164).
First publications, although a mere translation from the English, deserves to be mentioned, on account of the importance of its subject; for it comprised the most unexpected revelations on the mathematical culture among the Indians, namely, the *Treatises of algebra* that the English savants brought back from Calcutta. . . One may say, in praise of M. Terquem, that he was the one who opened the way to this field of researches in France, as soon as 1816.” 74 Although French scholars were already acquainted with British accounts of Sanskrit astronomy, previously published in the * Asiatic Researches*, Terquem’s merit, in Chasles’ eyes, was to highlight these new documents as being of paramount historical significance, for they supposedly made clear that “the Hindus had concerned themselves with abstract mathematics, knew algebra and had dealt with this science in an original way and with an unquestionable superiority over the Greek methods known to us.”75

In their account of the work of the British Indologists on Sanskrit mathematics, Hutton, and Terquem in his wake, heavily relied on Strachey’s extracts from the *Lilāvatī* and the *Bīja-ganita*, inserted in his *Observations*. They reproduced the same material Strachey had adduced on the series of topics upon which he had first laid stress: combinations, progressions, the mensuration of the circle and the sphere, quadratic equations, indeterminate problems of the second degree. As a rule, Terquem’s translation faithfully followed Hutton’s compendium. However, at certain junctures, interesting additions were made. In connection to the rule, found in the *Lilāvatī*, for approximating the ratio of the circumference of a circle to its diameter, *viz.* “multiply the diameter by 3927, then divide the product by 1250”, “a nearer approximation, than was known in Europe before Vieta”,76 Strachey higgledy-piggledy supplied four formulas, without any hint as to the way they may have been found.77 Among them, an approximate formula for computing the chord corresponding to a given arc in a circle of which the diameter and the circumference are known. Although the rule was first “expressed in long-winded sentences”,78 as Terquem later noted,79 Strachey wrote down the puzzling formula in algebraic notations: 

\[
\frac{4ac(D-c-a)}{2c^2-a(c-a)} = C,
\]

where \(D\) and \(c\) denote the diameter and the circumference of the circle, \(a\) the arc, and \(C\) the chord; “but, [Hutton later emphasized,] it does not appear how they have got this rule”.80 There, Terquem inserted an extensive footnote by Servois81 in which Bhāskara’s formula was tentatively derived from the Taylor’s series for the sine.82

As regards indeterminate analysis of the second degree, Hutton and Terquem fully relied on Strachey’s comparison of Diophantus’ and Bhāskara’s methods. To substantiate his views, Strachey quoted the 16th century French officer, artilleryman, and mathematician François-Joseph Servois (1767–1847), a former Constitutional priest and artillery officer, taught mathematics at the artillery schools. In addition to a few original contributions in projective geometry and a new mode of exposition of the differential calculus, he took part in the debates in Gergonne’s circle about Argand’s geometric representation of imaginary numbers, which he disparaged as “a geometric mask applied to analytic forms”.83

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75 Cf. Chasles (1863, p. 243).
76 Cf. Strachey (1808, p. 11).
77 However, as Hutton indicated, the first two formulas can be immediately derived from the geometrical figure, *viz.* \(\frac{1}{2}D - \frac{1}{2}\sqrt{D^2 - C^2} = V\) and \(2\sqrt{DV} - V^2 = C\), where \(D\) denotes the diameter of the circle, \(C\) the chord subtending an arc \(2a\) and \(V\) the versed sine of the half-arc \(a\). Still, the last two formulas remained intriguing for Strachey’s readers.
78 Cf. Terquem (1816, p. 262): “Toutes les règles de l’algèbre sont exprimées dans cet ouvrage de la même manière, en phrases très-longues, qui dans des cas compliqués, induisent aisément en erreur, et rendent ces règles difficiles à suivre; tandis qu’on comprend ces règles, pour ainsi dire, à la seule inspection, en se servant des notations en usage parmi nous.”
79 With hindsight, one may compare with Colebrooke’s translation of *Lilāvatī*, cf. Colebrooke (1817, p. 94): “art. 213. Rule: The circumference less the arc being multiplied by the arc, the product is termed first. From the quarter of the square of the circumference multiplied by five, subtract the first product: by the remainder divide the first product taken into four times the diameter: the quotient will be the chord.”
80 Cf. Hutton (1812, p. 159).
81 François-Joseph Servois (1767–1847), a former Constitutional priest and artillery officer, taught mathematics at the artillery schools. In addition to a few original contributions in projective geometry and a new mode of exposition of the differential calculus, he took part in the debates in Gergonne’s circle about Argand’s geometric representation of imaginary numbers, which he disparaged as “a geometric mask applied to analytic forms”.
82 Cf. Terquem (1816, pp. 265–266). Using the relation between the sine and the chord of an arc \(a\), *viz.* \(\sin a = \frac{1}{2} \text{chord } 2a\), Servois truncated the Taylor’s series for the sine after the first two terms and hence accounted for the Sanskrit formula.
question of the 6th book of Diophantus, viz. given two numbers \(a\) and \(b\), and two squares such that \(ax^2 - b = y^2\), to find two other squares for which the same holds. For instance, given 3 and 11, and a certain square \(5^2\), such that \(3.5^2 - 11 = 8^2\), another perfect square, “to find another square greater than 25 having this property”. In modern notations, as Terquem chose to put it, so goes Diophantus’ method. Let \(N + 5\) be the side of the sought square, then its square will be \(N^2 + 10N + 25\), the triple of which diminished by 11 leaves \(3N^2 + 30N + 64\), which must be a perfect square. Let its side be \(2N - 8\,\text{,} \) “one draws from this supposition (in forming the equation) \(N = 62\), and \(N + 5 = 62 + 5 = 67\); \(67^2 = 4489\); so 4489 is the required square.”\(^{83}\) Rather bluntly, without further comment, Strachey then contrasted the above with Indian algebraic methods:

In the Bija Ganita this problem is solved very generally and scientifically, by the assistance of another, which was unknown in Europe till the middle of the 17th century; and first applied to questions of this nature by Euler, in the middle of the 18th century – With the affirmative sign, the Bija Ganita rule for finding new values of \(ax^2 + b = y^2\), is this: Suppose \(ag^2 + b = h^2\) a particular case: find \(m\) and \(n\) such that \(an^2 + 1 = m^2\); then \(x = mg + nh\), and \(y = mh + ang.\(^{84}\)

It is to be noted that neither Hutton, nor consequently Terquem, did develop this rather cryptic allusion to the bhāvaṇā rules. They did not for instance rely on Strachey’s detailed analysis of the procedures these rules would involve, nor did they in the least mention his mathematically penetrating interpretation of the cākra-vāla in the light of the Euler–Lagrange theory of continued fractions. Although they quoted extensively from Strachey’s early Observations, they did not seem to be acquainted, or even aware, of his thorough investigations – later to be issued in print – into the indeterminate analysis of the Bija-ganita,\(^{85}\) apart from a few sparse remarks Strachey may have communicated to Hutton through one channel or another. While there is no question of continued fractions in the Observations, yet Hutton incidentally indicated that “it is probable that they [viz. the Hindus] have some system of continued fractions”,\(^{86}\) albeit without yielding Strachey’s grounds supporting this view. Relying on Burrow’s notes and Strachey’s translations, Hutton offered a description of the structure of Bhāskara’s algebraic treatise, and an overview of the content of each of its parts. Besides Bhāskara’s collection of problems of indeterminate analysis being clearly itemized in modern algebraic notation,\(^{87}\) another topic, particularly highlighted in Hutton’s (and Terquem’s) account, should be mentioned here, for it attracted the attention of many nineteenth-century European scholars. Hutton reported in a more detailed way than even Strachey in his later printed edition of the Bija-ganita, about the so-called “figure of the bride”.\(^{88}\) After quoting a passage from Strachey

\(^{83}\) Cf. Terquem (1816, p. 269). Terquem adds the commentary between parentheses. Given the first two squares \(5^2\) and \(8^2\), one takes as side of the first required square, any number exceeding 5, that is \(N + 5\). Then one forms the second square so that it equates \(3(N + 5)^2 - 11 = 3N^2 + 30N + 64\), that is a square whose side corresponds to 64 plus or minus a certain quantity. By a process of trial and error, one then hits upon \(2N - 8\), which yields a solution.

\(^{84}\) Cf. Strachey (1808, p. 11); see also Hutton (1812, p. 162) and Terquem (1816, pp. 269–270), in which Strachey’s passage above is reproduced almost identically.

\(^{85}\) As a matter of fact, Hutton received part of Strachey’s translations through Dalby, but it is not known whether he was sent the preface and the notes to the printed edition of 1813.

\(^{86}\) Cf. Hutton (1812, p. 174).

\(^{87}\) Cf. Hutton (1812, p. 165): “Several of them are not easy; and Mr. Burrow has sometimes given answers in the margin, done in the European manner.” See also Hutton (1812, pp. 173–174).

\(^{88}\) Cf. Strachey (1813, p. 54): “The Arabs call the 47th proposition of the first book of Euclid, ‘the figure of the bride’. I do not know why.”
on a Hindu rule amounting to the Pythagorean theorem, he added the following reflection of his own, supplemented with the exhibition of the very figure described:

In the margin of the original, as here annexed, is drawn a figure of four equal right-angled triangles, ..., exhibiting a new and obvious proof of the 47th of Eucl. 1; for here are the four equal right-angled triangles, which are equal to twice the rectangle of their two perpendicular sides; and which, together with the small square in the middle, being the square of the difference of those sides, make up the large square on the hypotenuse. ...And this may be considered as the Indian demonstration of the celebrated property of the sides of right-angled triangles, ...a property so much employed by their geometers, and so often referred to in their writings, by the name of “the figure of the bride”, and “the figure of the bride’s chair”, and “the figure of the wedding chair”; epithets which we may conjecture have been suggested by the above figure bearing some resemblance to a palanquin, or sedan chair, in which it is the usual practice, in that country, for the bride to be carried home to her husband’s house.

Terquem fully endorsed Hutton’s intimation that the figure would yield “the Hindu demonstration of this famous proposition”. This suggestion, presumably articulated here for the first time, namely before Colebrooke’s authoritative translation from the Sanskrit, later elicited a whole tradition of comparative analyses focusing on the role of diagrams in Sanskrit and Greek sources. On the whole, Hutton’s and Terquem’s reception of Strachey’s work on the Bija-ganita mainly emphasized the differences between Hindu algebra and Diophantus’ methods, as they were strikingly articulated in the following passage from the Observations, integrally reproduced in both accounts.

[The Bija-ganita] will be found to differ much from Diophantus’s work. It contains in Mr. Strachey’s opinion, which is highly probable, a great deal of knowledge and skill, which the Greeks had not; ...The arrangement and manner of the two works are as different as their substance: the one constitutes a regular body of science; the other does not: the Bija Ganita is quite connected and well digested, and abounds in general rules, which suppose great learning; the rules are illustrated by examples, and the solutions are performed with skill. Diophantus, though not entirely without method, gives very few general propositions, being chiefly remarkable for the dexterity and ingenuity with which he makes assumptions for the simple solutions of his questions. The former teaches algebra as a science, by treating it systematically; the latter sharpens the wit, by solving a variety of abstruse and complicated problems in an ingenious manner. The author of the Bija Ganita goes deeper into his subject, treats it more abstractedly, and more methodically, though not more acutely, than Diophantus.

In Strachey’s Observations however, this comparison is only tentatively submitted in a footnote as being “made from a view somewhat partial and superficial, [for] the Beej Gunnit is seen through a very defective medium”, viz. a late Persian translation. At that time, Strachey only intended to prompt “a good discussion” about the question “whether it is possible, and if possible, whether it is probable, that the Algebra of the Hindus is nothing more than a branch of Greek science long lost, but now restored?” Although persuaded that it is not, he nevertheless refrained from drawing any premature conclusions on this score.

90 Cf. Hutton (1812, p. 172).
91 Terquem refers to the figure as the “figure de la fiancée”, or the “figure de la chaise nuptiale”. Cf. Terquem (1816, p. 277): “Cette nouvelle démonstration du théorème de Pythagore est remarquable, en ce qu’on peut la regarder comme la démonstration indou de cette célèbre proposition.”
93 Cf. Strachey (1808, p. 5).
94 Cf. Ibid.
In this early stage of his reflections, he also left open the perplexing problem of one presumably common source for both Arabic algebra and Diophantus’ algebra, in spite of all their differences.\textsuperscript{95} Later on, his views on this topic would become more assured, being buttressed by a thorough mathematical account of what made the algebra of the \textit{The Bija-ganita} into a well-connected theoretical ensemble. Interestingly enough, the first reception of Strachey’s work in Britain and France left out these later more convincing arguments in support of his early suggestion about a probable Indian origin of algebra.

In 1817, Jean-Baptiste Joseph Delambre (1749–1822) reported on John Taylor’s translation of Bhāskara’s \textit{Lilāvati} before the Paris Académie royale des sciences. He noted that the ancient Hindus would “solve the indeterminate problems of the first degree, more or less as we do, \textit{thus showing} nothing but very ordinary.”\textsuperscript{96} Being interested in the usefulness of algebraic methods in astronomy, he did not highly esteem ancient Hindu science.\textsuperscript{97} In his \textit{Histoire de l’astronomie ancienne} (1817), he also considered Davis’s and Burrow’s work, as well as Strachey’s translation of the \textit{Bija-ganita}. He offered a somewhat more detailed account of the \textit{kuṭṭaka} and explained how it served to solve astral problems.\textsuperscript{98} He also briefly mentioned the method for solving indeterminate equations of the second degree.\textsuperscript{99} On the whole however, he downplayed Hindu algebraic methods.

4. Chasles’s conjecture

In February 1837, Chasles published an interesting note on the history of indeterminate analysis in the \textit{Journal de Liouville}. In this paper which grew out of extensive groundwork on Sanskrit sources, preparatory to the well-known Note XII of the \textit{Aperçu historique} (1837), he developed a subtle and complex argument, both mathematical and historical. His point was twofold. While claiming on the one hand, like Guglielmo Libri (1803–1869) before him, that Sanskrit formulas for solving the equation $Cx^2 \pm A = y^2$ were fundamentally identical with Euler’s, he endeavored to shed light on the peculiar way in which these formulas may have been obtained in Sanskrit mathematics. But on the other hand, in so doing, he also attempted to remedy what he deemed a significant omission in the received historical view from Colebrooke to Libri. His motivations for striving to restore some degree of historical continuity between Sanskrit and modern mathematics will be seen to be thoroughly connected to his broader agenda aiming at promoting pure geometry against analysis. More specifically, he was led to a fine geometrical interpretation of the Sanskrit algebraic rules, known as the \textit{bhāvanā} rules, as a consequence of his original reading of Brahmagupta’s rules for constructing rational quadrilaterals. However, while making the best of geometrical arguments, on the whole he barely touched the purely algebraic topic of \textit{cakravāla}.

As regards the work of the British Indologists, Chasles, who already knew about Strachey by Terquem, apparently became aware of the importance of Colebrooke through Libri’s first volume of the \textit{Histoire des mathématiques en Italie}. The first copies of Libri’s book had been handed out in Paris scholarly circles at

\textsuperscript{95} However, from Hutton’s to Terquem’s account, one may notice a gradual shift toward more positiveness. Cf. Hutton (1812, p. 175): “And it is not unlikely that there are old Hindu treatises, from which not only the Bija Ganita, but even the algebra of Diophantus, and that of the Arabs, may have been derived”. Cf. Terquem (1816, p. 280): “C’est de ces anciens ouvrages indous, dont l’existence est très-probable, qu’auront été tirées non-seulement la Bija Ganita, mais aussi l’Algèbre des Grecs et des Arabes”.
\textsuperscript{96} Cf. Delambre (1817b, p. 544).
\textsuperscript{97} Cf. Delambre (1817b, p. 545): “Si la science des Indiens est toute entière dans le livre dont on vient de lire l’extrait, on ne concevra guère comment ils auraient pu avoir une astronomie véritable et qui leur appartient.”
\textsuperscript{98} Cf. Delambre (1817a, p. 551): “Il y a des kutukas de plusieurs espèces: celui qu’on appelle fixé sert aux astronomes pour trouver le nombre de jours écoulés depuis le commencement du calpa [\textit{viz. the lifetime of the universe}, or \textit{4,320,000,000 years}]. . . . On voit que ce problème n’est pas d’une grande utilité, et, dans tous les cas, il ne convient qu’à l’Astronomie indienne.”
\textsuperscript{99} Cf. Delambre (1817a, p. 555): “Par la méthode du Jeist et du Canist, si l’on a deux sinus, il est aisé de trouver les autres, quand on connaît la nature du cercle; et ainsi par l’addition des \textit{sours}, on peut trouver la somme et la différence de l’arc, et son sinus peut être calculé.”
the end of 1835 already, before a fire destroyed the whole bookstore and delayed effective republication until 1838. On the other hand, a first version of Chasles’s own Aperçu historique was sent to the publisher also in 1835, but the process of impression “soon slackened, in particular by the study of the Indian works of Brahmagupta, whose real topic and whose special significance for the geometrical part had not yet been pointed out.” As for his contribution to the historiography of indeterminate analysis, Chasles drew in an innovative and original way on Libri’s most influential counterfactual pattern for conjectural history:

Two monuments of Indian algebra, Brahmagupta’s and Bhāscara Achārya’s treatises, were published in the current century, by MM. Colebrooke, Taylor and Strachey; and one must admit, in spite of all our occidental pride, that if these works had been brought to Europe sixty or eighty years earlier, their appearance, even after Newton’s death and during Euler’s lifetime, could have hastened the progress of algebraic analysis among us.

Libri fully endorsed Strachey’s point about Hindu algebraic methods being more general than those of Diophantus, and so did Chasles in their wake. However, Strachey mostly sidestepped geometry in his account of Bhāskara’s Bījā-ganita, and Colebrooke explicitly downplayed it in contrast to Hindu algebra. On the contrary, Chasles resolutely placed the emphasis on geometry. In the context of the Aperçu historique, he addressed Sanskrit sources with the same concerns in his mind as those infusing his more encompassing reflections on the generality of geometrical methods. In his view, both principles of duality and homography endowed modern projective geometry with no less generality, and definitely more simplicity and intuitiveness, than analytical procedures, for the transformations of figures codified by these principles paralleled and rivaled the transformations of formulas in analysis. While investigating the procedures of algebra, Chasles identified the main cause of the advantages these procedures were supposed to bring to geometry when applied to it. He also traced to its source “the introduction of that new element, that germ of further progress, the algebraic calculus.” Accordingly, Chasles’s historical notes originally purported to unfold the whole process from its presumed starting point, hence from Vieta and his literal

102 Cf. Libri (1838–1841, I, p. 126): “Deux monuments de l’algèbre indienne, le traité de Brahmagupta et celui de Bhāscara Achārya, ont été publiés, dans le siècle actuel, par MM. Colebrooke, Taylor et Strachey; et l’on doit avouer, malgré tout notre orgueil occidental, que si ces ouvrages eussent été apportés en Europe soixante ou quatre-vingts ans plus tôt, leur apparition, même après la mort de Newton et du vivant d’Euler, aurait pu hâter parmi nous les progrès de l’analyse algébrique.”
104 Cf. Chasles (1837b, p. 37): “ce grand analyste de l’école d’Alexandrie a montré beaucoup d’adresse et de génie; mais ces solution sont diverses, appropriées à des questions particulières, et ne mettent pas sur la voie des méthodes générales dont cette partie de l’analyse était susceptible.”
105 For a thorough account of Chasles’s reflections on the value of generality in the historiography of geometry, see the forthcoming work of Karine Chemla, cf. Chemla (forthcoming).
106 Cf. Chasles (1837a, p. 268): “Ces moyens que possède la Géométrie récente, de multiplier ainsi à l’infini les vérités géométriques, peuvent être assimilés aux formules et aux transformations générales de l’algèbre, qui donnent avec sûreté et promptitude la réponse aux questions diverses qu’on leur soumet: …ces moyens sont donc de véritables instrumens, que ne possédait point l’ancienne Géométrie, et qui font le caractère distinctif de la Géométrie moderne.”
107 Cf. Chasles (1837a, p. 196): “Mais au surplus, en réfléchissant sur les procédés de l’algèbre et en recherchant la cause des avantages immenses qu’elle apporte dans la Géométrie, ne s’aperçoit-on pas qu’elle doit une partie de ces avantages à la facilité des transformations que l’on fait subir aux expressions qu’on y introduit d’abord? …N’était-il pas naturel de chercher à introduire pareillement dans la Géométrie pure des transformations analogues, portant directement sur les figures proposées, et sur leurs propriétés?”
geometry. What apparently struck Chasles as a significant feature was that, on occasion, algebraic rules were proven by geometrical considerations. In his view, this fact sufficed to indicate that Hindu mathematicians had been acquainted with geometry, before they even conceived of their algebraic rules. One might have accounted for this, Chasles conceded, by supposing that they received geometry from the Greeks, while they created their algebra by themselves. “However, [he claimed,] one cannot believe that the Indians, a very ancient people, did not have a geometry before the Greeks”, (cf. Figure 1). According to this profession of faith, Chasles’s approach to Sanskrit sources branched off and explored a possibility hitherto unsuspected by mainstream historiography. He departed from both Strachey and Colebrooke on at least two significant scores, namely in focusing on geometry rather than algebra, and in shifting his attention from Bhāskara to Brahmagupta. On the whole, Colebrooke allegedly misinterpreted the geometrical part of Brahmagupta’s, and even Bhāskara’s, work, because he was looking for “elements of geometry” of some kind, and therefore only paid heed to “a few elementary and primary propositions, upon which all of Hindu science [presumably] rested”. This bias in return induced the view that “they (the Hindus) cultivated Algebra much more, and with greater success, than Geometry; as is [purportedly] evident from the comparatively low state of their knowledge in the one

109 For Chasles, as Karine Chemla underscores (see Chemla, forthcoming), one of the main properties of algebraic formulas is to allow the geometric to deal with various geometrical configurations at one blow, without distinguishing cases as in Greek geometry. Accordingly, Chasles’s emphasis on Vieta in the first elaboration of his historical notes appears justified since Vieta may arguably be held as the first initiator of literal algebra, in which letters stand for both variables and parameters – an essential feature accounting for the above key characteristic of formulas.

110 Ibid.


112 Cf. Chasles (1837a, p. 419), see also Chasles (1870, p. 98).
and the high pitch of their attainments in the other.”\textsuperscript{113} However, Chasles much later pointed out, “the mismatch – mostly – between the perfection of Brahmagupta’s fragment of algebra and the apparent inferiority of the geometrical part, prompted a new scrutiny of the Hindu work.”\textsuperscript{114} Chasles then turned to Colebrooke’s translation of the mathematical tracts of Brahmagupta’s \textit{Brāhma-sphuṭa-sidhānta}, and mainly focused on that part registered by his British predecessor as chapter XII, “Arithmetic (\textit{Ganîta})”, section IV, “Plane figure”.\textsuperscript{115} Therein, he highlighted some most valuable propositions which Colebrooke had overlooked, “probably because they stood as so many enigmas, due to their laconism and the obscurity of their statement, in each of which certain conditions forming the hypothesis are always implied”.\textsuperscript{116} In spite of their terseness, Chasles claimed and set out to show that these propositions, albeit “at first appearing unlinked and as if thrown haphazardly”,\textsuperscript{117} proved on closer inspection to be organized into a single unified geometrical theory, the theory of cyclic quadrilaterals. Far from forming elements of geometry, these propositions would make sense as a consistent whole, provided they were correctly and insightfully interpreted. They would fit into one another so as to solve one single problem: \textit{viz.} to construct a quadrilateral to be inscribed into a circle, so that its area, its diagonals, its peripherals and various other lines, as well as the diameter of the circumscribed circle, be expressed in rational numbers.\textsuperscript{118} This sample of geometry bears witness, “if not to a very extensive knowledge, at least \textit{[in Chasles’s view,] to a certain skill in geometry, and a habit of computation; [hence] in this respect, \textit{[it proves to be] in the algebraic spirit of the Hindus.”}\textsuperscript{119}

Furthermore, Chasles reversed the historical order of prominence. Indeed, Colebrooke had held that Bhāskara was more knowledgeable than Brahmagupta, whose propositions he rectified on occasion, and Strachey had shown how Bhāskara had successfully enriched Brahmagupta’s methods in indeterminate analysis so as to achieve \textit{cakravāla}. But, owing to his insight into the presumably true meaning of Brahmagupta’s geometry, Chasles made clear that in some instances, Bhāskara did not understand Brahmagupta’s rules anymore.\textsuperscript{120} Besides, while looking back on his early attempts in his 1870 report on the progress of geometry, Chasles considered this fragment of geometry on cyclic quadrilaterals on a par with the fragment of algebra on indeterminate analysis. “Both proved \textit{[pace Colebrooke and Strachey,] that algebra and geometry had been cultivated as successfully, from the theoretical standpoint, that is from the standpoint of science proper, by the old Hindu people, and we may add, in an original way, superior to Greek science.”\textsuperscript{121}

\textsuperscript{113} Cf. Colebrooke (1817, p. xv). In the \textit{Aperçu historique} (cf. \textit{Ibid.}), Chasles quoted Colebrooke’s statement and connected his implicit preconception of what proper geometry should look like, with his comparative ranking of algebra and geometry in Sanskrit sources.

\textsuperscript{114} Cf. Chasles (1870, p. 99): “Cette considération \textit{[the notion of a Hindu decline through ages],} et la discordance surtout qui avait lieu entre la perfection du fragment d’algèbre de Brahmeaupûta et l’infériorité apparente de la partie géométrique, provoquaient à un nouvel examen de l’ouvrage hindou.”

\textsuperscript{115} Cf. Colebrooke (1817, pp. 305–311).

\textsuperscript{116} Cf. Chasles (1870, p. 99).

\textsuperscript{117} Cf. Chasles (1837a, p. 419).

\textsuperscript{118} Chasles’ interpretation of Brahmagupta’s propositions on quadrilaterals, reconstructed in the light of archival sources, will be the topic of a separate forthcoming contribution.

\textsuperscript{119} Cf. Chasles (1837a, p. 420): “Ces questions dénotent, sinon un savoir très-étendu, du moins une certaine habileté en Géométrie, et une habitude du calcul. Sous ce rapport, elles sont dans l’esprit algébrique des Hindous.”

\textsuperscript{120} Chasles (1837a, p. 420): “L’ouvrage de Bhâscara n’est qu’une imitation très-imparfaite de celui de Brahmeaupûta, qui y est commenté et dénaturé. . . . les propositions les plus importantes de Brahmeaupûta, relatives à la théorie du quadrilatère inscriptible au cercle, y sont omises, ou énoncées comme \textit{inexactes.} Ce qui montre que Bhâscara ne les a pas comprises.”

\textsuperscript{121} Cf. Chasles (1870, p. 100): “Ce fragment de géométrie n’a rien de plus pratique que le fragment d’algèbre qui renferme la théorie des équations indéterminées du premier et du second degré. Les deux prouvent que l’Algèbre et la Géométrie avaient été cultivées avec le même succès, au point de vue théorique, c’est-à-dire de la science proprement dite, par l’ancien peuple hindou, et nous pouvons ajouter, d’une manière originale et supérieure à la science grecque.”
Obviously then, Chasles recounted with hindsight, the question arose whether both fragments were related and, if so, in what way.

Is it by chance that the theory of indeterminate equations of the second degree and that of the quadrilateral liable to be inscribed in a circle, are gathered in Brahmagupta’s work? One may think so, since both questions, the one of pure analysis and the other of pure geometry, do not seem to come into contact in any way. But there exist secret connections between all parts of mathematics; therefore, there was a need there for reflection and examination, all the more so as one knew by Luca of Burgo [viz. Luca Pacioli] that the formulas Leonardo of Pisa gave in his Treatise of the square numbers, for solving the equation \( x^2 + y^2 = A \), were demonstrated there by considering geometrical figures.

The study of that question has shown that Brahmagupta’s algebraic formulas can be effectively deduced from the constructions of the geometrical question.122

This is what the Note published in the Journal de Liouville was all about. Like Strachey two decades earlier, Chasles endorsed Lagrange’s historical account of the development of indeterminate analysis in modern times, albeit only partially, only for the stretch ranging from Fermat to Euler.123 However, as seen above, the British Indologist had read Bhāskara’s algebraic methods in the light of Lagrange’s theory of continued fractions, which purportedly completed Euler’s previous contributions while integrating them into one single consistent mathematical theory.124 By contrast, Chasles clearly shifted the emphasis from continued fractions to Euler’s formulas themselves, disentangled from Lagrange’s compellingly unified interpretational framework. On this occasion, he drew on a very general remark by Libri, rather than on Strachey’s mathematically more accurate reading. “In Brahmagupta’s and Bhāskara’s works [Libri had contended,] one finds that way to deduce from one solution of an indeterminate equation of the second degree with two unknowns all the other solutions: this analysis, which we owe to Euler, was known in India for more than ten centuries.”125 And Libri respectively pointed to Brahmagupta’s “chapter XVIII (Cuttaca)” in Colebrooke’s translation and to chapter VI of Euler’s algebra in the French translation with Lagrange’s additions. Chasles went further and substantiated Libri’s merely indicative statement with mathematical particulars excerpted from the works themselves. On the one hand, he extracted Euler’s formulas for solving

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122 Cf. Chasles (1870, p. 100): “Est-ce fortuitement que la théorie des équations indéterminées du second degré et celle du quadrilatère inscriptible au cercle se trouvent réunies dans l’ouvrage de Brahmagupta? On pourrait le croire, puisque ces deux questions, l’une de pure Analyse et l’autre de pure Géométrie, ne paraissent pas comporter de points de contact. Mais il existe des liens secrets dans toutes les parties des Mathématiques; il y avait donc lieu ici à réflexion et examen, d’autant plus que l’on savait par Luc as de Burgo que les formules données par Léonard de Pise, dans son Traité des nombres carrés, pour la résolution de l’équation \( x^2 + y^2 = A \), y étaient démontrées par la considération des figures géométriques.

L’étude de la question a fait reconnaître qu’effectivement les formules algébriques de Brahmagupta se pouvaient déduire des constructions de la question géométrique.”

123 See Chasles (1837b, p. 38), where Chasles refers to Lagrange’s remarks on the history of indeterminate analysis in §VIII of his Additions to Euler’s algebra, cf. Lagrange (1774, p. 157). The comments on the respective contributions of Brouncker, Wallis, Ozanam, Fermat and Euler are directly taken from Lagrange.

124 Lagrange acknowledged – and so did Chasles in his wake – that Euler had first grasped the importance of the so-called Pell’s equation \( Cx^2 + 1 = y^2 \) for solving the more general equation \( Cx^2 \pm A = y^2 \), to which all indeterminate equations of the second degree could be brought back. However, Euler had dealt with Pell’s equation in the context of arithmetical investigations for which he devised his well-known algorithm of continued fractions, see Euler ((1759) 1767). On the other hand, he obtained general formulas for solving \( y^2 = Cx^2 + A \), by different methods developed within an independent algebraic framework, see Euler ((1759) 1764) and Euler (1770, chapter VI and VII). Lagrange unified both strands and introduced the theory of continued fractions as an essential part of indeterminate analysis, cf. Lagrange (1774, p. 6): “La Théorie des fractions continues est une des plus utiles de l’Arithmétique, où elle sert à résoudre avec facilité des Problèmes qui, sans son secours, seraient presque intraitables; mais elle est d’un plus grand usage encore dans la solution des Problèmes indéterminés, lorsqu’on ne demande que des nombres entiers.”

the equation $y^2 = Cx^2 + A$ from the “analytical considerations” \footnote{Cf. Chasles (1837b, p. 39). Although Chasles did not give any precise reference to Euler’s work, he hinted at a passage of Lagrange’s Additions to Euler’s algebra, in which a paper by Euler is mentioned, De resolutione formularum quadraticarum indeterminatarum per numeros integros (1759/1764), published in the IXth volume of the Saint Petersburg Novi Commentarii. In this paper, Euler’s formulas for solving the equation $y^2 = Cx^2 + A$ first appear in the Scholion to Problem 3, cf. Euler (1759/1764, p. 16). They are deduced from more general formulas Euler first obtained to solve another problem, namely to find the values of $x$ which make the radical $\sqrt{ax^2 + bx + c}$ rational. Given a first solution $b^2 = a^2 + \beta a + \gamma$, Euler assumes that any further solution $y^2 = ax^2 + bx + \gamma$ can be written $x = a + mz$, $y = b + nz$. By substitution, one obtains $z = \frac{2a(a + \beta m) + \beta n}{n^2 - am^2}$, which eventually yields general formulas for $x$ and $y$. So as to get rid of the fractions and reach integral solutions, one obviously has to adduce the further condition on the denominator that $n^2 - am^2 = 1$. Hence one grasps the connection Euler first saw between the indeterminate equation $y^2 = ax^2 + \gamma$ and Pell’s equation $n^2 - \alpha m^2 = 1$. In his 1770 algebra, Euler nevertheless indicated a shorter way to obtain directly his formulas in integers without having to derive them from fractional expressions, cf. Euler (1770, chapter VI, art. 85–86, pp. 102–103). However, this way merely amounted to an ex post facto algebraic verification which partially presupposed the formulas in question and thus, as Lagrange emphasized, did not disclose their rationale. Cf. Lagrange (1774, §VIII, p. 158); “d’un côté, la méthode d’Euler conduit naturellement à des expressions fractionnaires lorsqu’il s’agit de les éviter, et de l’autre on ne voit pas clairement que les suppositions qu’on y fait pour faire disparaître les fractions soient les seules qui puissent avoir lieu.”} by which Euler had obtained them in the first place. On the other hand, he transcribed in modern algebraic notations a pair of rules formulated by Brahmagupta in section VII, “Square Affected by Coefficient”, of Colebrooke’s English translation, which he quoted extensively in a footnote.

\textit{First rule.} \footnote{This rule corresponds to art. 65–66, cf. Colebrooke (1817, p. 363).} For solving the equation $Cx^2 + 1 = y^2$, one takes a system of roots of the equation

$$C x^2 \pm A = y^2,$$

where $A$ is indeterminate. Let these roots be denoted $l$ and $g$, so that $C l^2 \pm A = g^2$, the roots of the proposed equation will be

$$y = \frac{C l^2 + g^2}{A}, \quad \text{and} \quad x = \frac{2l}{A}.$$

By replacing $A$ by $g^2 - C l^2$, and by making $l = 1$, one will have precisely the expressions found by Fermat, Brouncker, etc. \footnote{Namely by taking $m = \frac{4}{l}$, one obtains $x = \frac{2m}{m^2 - A}$ and $y = \frac{m^2 + C}{m^2 - A}$.}

\textit{Second rule.} \footnote{This rule corresponds to art. 68; cf. Colebrooke (1817, p. 364).} For solving the equation $Cx^2 \pm A = y^2$, when one knows a first system of roots $L$, $G$ for it, one takes a system of roots of the equation

$$C x^2 + 1 = y^2;$$

let these roots be denoted $l$ and $g$; the general expressions for the roots of the proposed equation will be

$$x = L g + l G,$$

$$y = C l l + G g.$$

\textbf{[About] both these rules [Chasles eventually stressed that they were} identical with Euler’s solution. \footnote{Cf. Chasles (1837b, pp. 39–40).}
Chasles obviously chose the letters $L, l$ and $G, g$ on purpose to recall the “least roots” and the “greatest roots” of the so-called bhāvanā rules. At this juncture, he took a bold step way beyond Libri. With great sagacity, he unearthed specific formulas, attributed to Fibonacci,\textsuperscript{131} which, he suggested, might help bridge the historical gap between Brahmagupta’s rules and Euler’s formulas.

But it seems that one has always committed an omission on this score [viz. as regards the hitherto received account of the history of indeterminate analysis], an omission which it is all the more appropriate to rectify at this point, when it comes to Indian analysis, as it precisely concerns a solution which seems to us to derive from Hindu works – a solution which would have made up for these works, and would have put at once the geometers who may have known them on the track of the discoveries reserved for Euler. We are talking about the few questions of indeterminate analysis solved by Fibonacci (commonly called Leonardo of Pisa) in his treatise of algebra, an original piece of work which remained in the state of a manuscript much to the regret of the geometers.\textsuperscript{132}

Fibonacci’s formulas, as Chasles identified them on the basis of scarce textual evidence ultimately drawn from treatises by the Renaissance algebraists Luca Pacioli (1445–1517) and Girolamo Cardano (1501–1576), arose as the solution of the following problem: “How to derive further rational solutions from a first system of roots $x, y$ of the equation $x^2 + y^2 = A$?” Assuming two arbitrary numbers $a$ and $b$, the sum of whose squares is a square, viz. $a^2 + b^2 = c^2$, – which can be done in infinitely many ways,\textsuperscript{133} – then, Chasles explained, from both equations

\[x^2 + y^2 = A,\]
\[a^2 + b^2 = c^2,\]

one derives the general expressions for the roots of the proposed equation

\[x' = \frac{ay + bx}{c},\]
\[y' = \frac{by - ax}{c}.\]

\textsuperscript{131} Most of the biographical information (then available) about Leonardo Pisano (1170–1240), known as Fibonacci, was drawn from the prologues to the Liber Abaci (1202), cf. Sigler (2002, pp. 15–16), and the Liber quadratorum (1225), cf. Sigler (1987, p. 3), in particular his connection with the court of Frederick II of Sicily. However, before the very first sources to which Cossali had access were known, Montucla had misdated Fibonacci’s work to the fourteenth century, which apparently ignited the controversy between them. Cf. Montucla (1799, p. 715): “L’impitoyable M. Cossali […] me poursuit à chaque page, avec une sorte d’animosité qui ne serait pas plus grande, quand je serais coupable de quelque crime de lèze nation Italiene, ou que je l’aurais blessé personnellement.” For a detailed account of the context of the encounter between Fibonacci and Frederick II (1194–1250), cf. Kantorowicz (1957).

\textsuperscript{132} Cf. Chasles (1837b, p. 42): “Mais il paraît qu’on a toujours commis sur ce point une omission, qu’il est d’autant plus à propos de réparer ici, en parlant de l’analyse indienne, que cette omission porte précisément sur une solution qui nous paraît dériver des ouvrages hindous; solution qui aurait suppléé ces ouvrages, et aurait mis aussitôt sur la voie des découvertes réservées à Euler, les géomètres qui en auraient eu connaissance. Nous voulons parler de quelques questions d’analyse indéterminée, résolues par Fibonacci (appelé communément Léonard de Pise) dans son traité d’algèbre, ouvrage original, resté manuscrit au grand regret des géomètres.”

\textsuperscript{133} For the sake of his later argument, Chasles incidentally indicated here that one may for instance take arbitrarily the first number $a$, and then fix the second number $b = \frac{1}{2} (\frac{a^2}{m} - n)$, so that $a^2 + b^2 = a^2 + \frac{1}{4} (\frac{a^2}{m^2} - n)^2 = \frac{1}{4} (\frac{a^2}{m^2} + n)^2$. These formulas for the sides of a rectangular triangle are those one finds in Brahmagupta, see Rule 35 in chapter XII, “Arithmetic (Gaṇita)”, section IV, “Plane figure” in Colebrooke’s translation, cf. Colebrooke (1817, p. 306).
These formulas were to play a pivotal role in the historiographical construction which Chasles shaped by relying on both modern and ancient sources. Like Libri, Chasles used the works of Pietro Cossali, Jean-Étienne Montucla and of course Colebrooke. Cossali had first mentioned a lost manuscript on square numbers by Fibonacci, which had been found by Targioni Tozzetti in the Magliabechian library at Florence, together with a copy of an abacus treatise by the same author. Although neither Cossali, nor any other scholar up to Chasles, had been able to recover that manuscript, Cossali was commonly held at that time to have “successfully” restored the lost treatise out of fragments of Luca Pacioli’s *Summa de arithmetica, geometria, proportioni et proportionalitā* (1494). Chasles may have picked out Fibonacci’s formulas from Cossali’s version of the problem originally dealt with in the *Liber Quadratorum*. But he also checked Pacioli’s *Summa I*, 4, 9, and found there a piece of information about the way Fibonacci presumably obtained the above formulas that struck a chord with him. In combination with what he had stumbled upon in his parallel investigations into Brahmagupta’s rules on quadrilaterals, all added up and crystallized into a conjecture.

As for the attribution of these formulas to Leonardo Pisano, Chasles deferred to the testimony of both Pacioli in his *Summa* and Cardano in his *Practica arithmetice* (1539). Futhermore, he emphasized the analogy between Fibonacci’s formulas and those of Brahmagupta and Euler, of which the former “are only a particular case”. Indeed, by simply rewriting both equations assumed by Fibonacci according to the Sanskrit pattern for indeterminate equations of the second degree,

\[
\begin{align*}
    x^2 + y^2 &= A, \\
    a^2 + b^2 &= c^2, \\
    y^2 &= -x^2 + A, \\
    b^2 &= -a^2 + c^2,
\end{align*}
\]

so that both equations have the same “nature” (*prakṛti*) \( N = -1 \), and respectively \( A \) and \( c^2 \) as “interpolators”, one can apply Brahmagupta’s procedure for the *bhāvanā* rule (as seen above):

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135 Leonardo’s *Liber Quadratorum* was only edited in 1854 by Baldassare Boncompagni. See below for an assessment of Chasles’s conjecture in view of what we now know of Fibonacci’s work.
136 On this score, Chasles took his cue from Montucla and Colebrooke. Cf. Montucla (1799, p. 715): “Léonard de Pise avait aussi écrit un traité intitulé De’ numeri quadrati, qui ne se trouve plus, mais que M. Cossali restitue d’après les divers fragments qui s’en trouvent dans Lucas de Burgo.” See also Colebrooke (1817, p. lii): “No diligence of research has, however, regained any trace of the volume which contained Leonardo’s treatise on square numbers: the library, in which it was seen, having been dispersed previously to Cossali’s inquiries.”
137 Cf. Chasles (1837b, p. 43). Chasles’s assessment of Cossali’s attempt stems from Colebrooke, see Colebrooke (1817, p. lvii): “Besides his great work on arithmetic and algebra, Leonardo was author of a separate treatise on square numbers. Reference is formally made to it by Paciolo [sic], who drew largely from this source. … The directions for the solution of such problems being professedly taken by Paciolo chiefly from Leonardo, and the problems themselves which are instanced by him being probably so, it can be no difficult task to restore the lost work of Leonardo on this subject. The divination has accordingly been attempted by Cossali, and with a considerable degree of success.”
140 The “*distinctio prima, tractatus quartus, articulus nonus*” of Pacioli’s *Summa* reads: “Damme questa. 4 via 4 fa 16 e 5 via 5 fa 25, che gionti questi loro doi quadrati insiem fano 41. Or trovame doi altri numeri che li loro quadrati similmente gionti insiem facino 41 e non siano nui dil aposti.”, cf. Pacioli (1494, 17v–18r).
141 For Pacioli, see Pacioli (1494, 18r): “… e mai falla questa regola. La quale donde ella preceda Leo Pi. nel tractato che fa de quadrati numeris la dimostra per via de figure geometriche le quali demonstration spero in questo adurle.” And for Cardano, see Cardano (1539, Caput 66, Quaesto. 44): “*questiones sunt Leonardi Pisaurenensis viri clarī*.”
142 Cf. Chasles (1837b, p. 44).
and thus obtain Fibonacci’s formulas. These in return strikingly contrast, Chasles pointed out, with Diophantus’s solution to the same problem\textsuperscript{143} – a solution which, couched in modern algebraic notations, would yield the following formulas\textsuperscript{144}:

\[
\begin{align*}
x' &= \frac{(n^2 - 1)x + 2ny}{n^2 + 1}, \\
y' &= \frac{2nx - (n^2 - 1)y}{n^2 + 1}.
\end{align*}
\]

As they contain only one indeterminate \(n\) and do not make any use of the auxiliary equation \(a^2 + b^2 = c^2\), these formulas, Chasles emphasized, “are not appropriate to solving the problem in integers”.\textsuperscript{145} However, acknowledging the mere fact of this discrepancy between Diophantus’s methods and Fibonacci’s formulas would not suffice to ascribe the latter to a Hindu origin. Something more decisive was required. An incidental remark put Chasles on this track. While trying to make sense of Brahmagupta’s rules on quadrilaterals, through pages of preparatory notes,\textsuperscript{146} Chasles hit upon a completely unexpected way to derive Fibonacci’s formulas geometrically. This realization apparently prompted the opening of a new front in his campaign for the primacy of pure geometry. It suggested that geometry may also have led the way to significant results in indeterminate analysis. Besides, Chasles had noticed a turn of phrase in the relevant passage of Luca Pacioli’s \textit{Summa} that seemed to corroborate his interpretation. About the method for solving the equation \(x^2 + y^2 = A\), Pacioli indeed reported that Fibonacci had proved his solution “by the consideration of geometrical figures”,\textsuperscript{147} as Chasles translated Pacioli’s Italian phrase “\textit{per via de figure geometriche}”.\textsuperscript{148} Chasles then conjectured that Fibonacci’s formulas may have originated in India where they were demonstrated geometrically, then transmitted from the Hindus to the Arabs, and eventually handed down by the latter to Fibonacci who introduced Arabic mathematical knowledge in Europe. Interestingly, Chasles found

\textsuperscript{143} This problem occurs in Book II of Diophantus’s \textit{Arithmetica}. It is registered as problem X in Bachet’s edition, cf. Bachet de Méziriac (1621, pp. 89–90): “\textit{Datum numerum qui ex duobus componitur quadratis, in alios quadratos partiri}.” Tannery labels it as problem IX, book II, cf. Tannery (1893, p. 93): “\textit{Datum numerum, qui sit summa duorum quadratorum, partiri in alios duos quadratos}.”

\textsuperscript{144} Diophantus unfolds the whole procedure in the particular case of a numerical example \(4 + 9 = 13\). Transcribed in algebraic notation, it corresponds to the following sequence of steps. The problem is to break up the sum of two squares \(x^2 + y^2 = A\) into two further squares. Assuming that the sides of these further squares are of the form \(x + \alpha\) and \(n\alpha - y\), one sets the sum of the corresponding squares \((x + \alpha)^2 + (n\alpha - y)^2 = A\), from which one easily infers the expression for \(\alpha = \frac{2(ny - x)}{n^2 + 1}\). By replacing \(\alpha\) by this latter expression, one obtains Chasles’s formulas – except for the sign of the second root, because Chasles slightly departs from Diophantus in the first place by assuming \(y - n\alpha\) as the side of the second required square, instead of \(n\alpha - y\), as Diophantus did.

\textsuperscript{145} Cf. Chasles (1837b, p. 44).

\textsuperscript{146} These notes are still extant among Chasles’s papers held at the Paris \textit{Académie des Sciences}. Their inventory and exploitation will be reserved for forthcoming contributions.

\textsuperscript{147} Chasles stresses this point in both the Liouville paper and in the \textit{Aperçu}, cf. Chasles (1837b, p. 43) and Chasles (1837a, p. 442).

\textsuperscript{148} Cf. Pacioli (1494, p. 18). Chasles also pointed out that in spite of all his knowledge of Pacioli, Colebrooke apparently overlooked this passage, as well as the analogy between Fibonacci’s and Brahmagupta’s formulas, cf. Chasles (1837b, p. 43).
further evidence for assuming the Hindu origin of Fibonacci’s methods in the graphical disposition of numbers and the way arithmetical operations were to be processed, as was attested by both Pacioli and Cardano.

The way Luca of Burgo and Cardano arrange the known quantities on the paper, so as to carry out the computation of the sought roots, bears the greatest resemblance to the way of the Indians, and denotes the origin of their method. The Indians place both given roots $x, y$ next to one another on a horizontal line; and below, on a second horizontal line, they place both roots $a, b$ of the auxiliary equation [viz. $a^2 + b^2 = c^2$], so that $x$ and $a$ stand on a vertical line, and $y$ and $b$ on a second vertical line. Then they multiply by one another the quantities $x$ and $a$ that stand on the first vertical line, then the two others. They subtract both products, and divide the difference by $c$; and this is one of the roots. To form the other, they proceed to pairwise cross-products for the four numbers, whose sum, divided by $c$, yields the second root. The Italian geometers proceed in the same way, except that they place both roots $x$ and $a$ next to one another, and below both $y$ and $b$. Like the Indians, they use the expression *cross-product* [l’expression de multiplication *en croix*].

The close description of the Hindu procedure is modeled upon Colebrooke’s account of the *bhāvanā* rules. As regards the Italian algebraists, neither Pacioli, nor Cardano did prove the formulas credited to Fibonacci, but they showed how the procedure was to be carried out in a variety of examples. In the relevant passages of their printed works, diagrams occur in the margin which bear witness to its successive steps (cf. Figure 2). In his *Summa* I, 4, 9 for instance, Pacioli considers the particular case $4^2 + 5^2 = 41$, with the auxiliary equation $3^2 + 4^2 = 5^2$. The way the operations are to be processed is then stipulated down to the very details. One writes down 4 and 5 in a vertical line, as well as 3 and 4, also vertically to the left. One then proceeds to the multiplications both horizontally and “crosswise” [multiplicationi in croce], as shown in the first diagram on the left in Figure 2. The horizontal products are subtracted, while the cross-products are added, then one divides both this difference $20 - 12 = 8$ and this sum $16 + 15 = 31$, by 5. The results are the roots. If the numbers are spatially laid out in the alternative order with both pairs twisted, as for instance in the second diagram on the left in Figure 2, then, Pacioli emphasizes, the multiplications certainly yield different results, hence eventually different roots, and yet these roots “nonetheless solve the problem” [non dimeno solvano al thema]. In chapter 66, Question 44 of his *Practica arithmetice*, Cardano also developed a similar example with the same kind of graphical arrangement for the partial products. Chasles was struck

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149 Cf. Chasles (1837b, p. 45). Although relying on Pacioli in his account of the procedure leading to Fibonacci’s formulas, Cossali does not reproduce the graphical disposition of the partial products, cf. Cossali (1797–1799, I, pp. 119–120). Therefore Chasles went back to Pacioli’s *Summa*.

150 Cf. Pacioli (1494, p. 18).
by the fact that in both Italian and Sanskrit sources, the schematism of the whole arithmetical procedure was the same.

In the *Aperçu historique*, Chasles incidentally recalled that Vieta was the first to prove Fibonacci’s formulas in *Zetetica* IV 2,\(^{151}\) and that “his demonstration was analytical”,\(^{152}\) while shortly afterwards the Scottish mathematician Alexander Anderson set out to prove Diophantus’s formulas by means of geometrical considerations in the Euclidean way.\(^{153}\) The question therefore naturally arose whether the ancient Hindu mathematicians may have ‘proved’ their highly valuable results in indeterminate analysis, and if so, in what way. Chasles’s stance on this score was two-sided. One the one hand, he unambiguously stated that “Indian works [did] not contain any demonstration”,\(^{154}\) namely none patterned after the Euclidean canon. However, he noted, this indisputable fact wrongly induced some authors, biased in favor of the Greeks, to “ascribe the analytical discoveries of the Hindus to some chance encounters, only prompted by isolated attempts and made without any method nor intelligence”.\(^{155}\) While discarding this view as ungrounded, Chasles claimed on the contrary that Hindu works contained “the vestiges of a science which, being cultivated for a long time, had achieved, within certain limits, a great perfection”.\(^{156}\) This conviction most probably grew from his thorough work on Brahmagupta’s theory of quadrilaterals, contained in propositions XII 21 to 38 of Colebrooke’s translation. None of these propositions would be proved in Sanskrit sources, at least according to the received standards. However, Chasles showed how they were to be organized into a consistent whole and connected so as to make mathematical sense. He explained how, by their being exactly fitted to one another, they could convey mathematical knowledge with a certain stamp of perfection.\(^{157}\) Chasles’s manuscript notebooks testify to the process by which he came to envision the very possibility of solving indeterminate equations of the second degree in a purely geometrical way, while working out his overall interpretive reading of Brahmagupta’s propositions. By connecting propositions XII 28 and 38, Chasles made clear how certain presumably cyclic quadrilaterals with perpendicular diagonals (see Figure 3, middle) could be obtained by a definite procedure, out of two arbitrary rectangular triangles in numbers \(T = (a, b, c)\) and \(T' = (a', b', c')\) taken as generators (see Figure 3, left). He then successfully showed that in quadrilaterals of this type, all significant lines could be rationally expressed in terms of

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\(^{151}\) Cf. Viéta (1646, pp. 62–63): “*Invenire numero duo quadrata, aequalia duobus aliis datis quadratis.*”

\(^{152}\) Cf. Chasles (1837a, p. 442).

\(^{153}\) Cf. Anderson (1619, pp. 17–21).

\(^{154}\) Cf. Chasles (1837b, p. 45).

\(^{155}\) *Ibid.*

\(^{156}\) *Ibid.*

\(^{157}\) A detailed account of Chasles’s reading of Brahmagupta is reserved for a forthcoming contribution.
the sides of those triangles assumed as generators. In so doing, he noticed an interesting property of the perpendicular $CF$ and the segment $FD$ in the quadrilateral $ABCD$. Being the sides of a rectangular triangle $CFD$ with hypotenuse $CD$, previously proved $= bc'$, these lines could be rationally expressed as follows:

$$CF = \frac{b}{c}(ab' + ba'), \quad FD = \frac{b}{c}(bb' - aa').$$

Insofar as they contain neither the quantity $c'$, nor consequently the side $CD = bc'$, “these expressions [Chasles concluded,] would thus remain rational, even though the side CD itself were not so. Therefore the lines $CF$, $FD$ provide a geometrical solution to the problem of decomposing a given number (square or not) in two square numbers, knowing a first solution to the question.” As clearly shown in the series of diagrams from the notebooks, Chasles insightfully identified the geometrical way to solve the indeterminate equation $x^2 + y^2 = A$, by focusing on a particular motif encased in the figure of the quadrilateral, namely the triangle $ACD$ with both perpendiculars $OD$ and $CF$ (compare the middle and right diagrams in Figure 3). He then extracted the following “more direct and more elementary method” from the rather intricate theory of quadrilaterals. His starting point was a geometrical reformulation of the problem of indeterminate analysis.

**Question.** Knowing a first system of roots $x$, $y$ for the equation $x^2 + y^2 = A$, it is asked to find a further system of rational roots.

**Geometrical solution.** Assuming a rectangular triangle $COD$, in which the sides around the right angle are $OC = x$, $OD = y$, the question will be to construct another rectangular triangle $CFD$ on the hypotenuse $CD$, whose sides $CF$, $FD$ will be rational.

His solution will be seen to result from his thorough acquaintance with Brahmagupta’s rules XII. 21 to 38, which he cunningly used so as to trace a path toward his goal. In the quadrilateral configuration, both solutions, the one given and the one sought, were instantiated as two rectangular triangles $COD$ and $CFD$ with the same (rational) hypotenuse $CD$. Both these triangles had been obtained by taking perpendiculars in another triangle $CAD$, previously available in the quadrilateral configuration. Chasles’s line of reasoning then consisted of two steps: firstly identifying the necessary and sufficient conditions for ensuring that the sought sides $CF$, $FD$ be rational, and secondly fulfilling these conditions by geometrical constructions out of the given rational sides $OC$, $OD$. Assuming that the problem is solved, hence that both triangles $COD$ and $CFD$ are already drawn (cf. Figure 4), one recovers the triangle $CAD$ by extending both lines $CO$ and $DF$ so that they intersect in point $A$. In that triangle $CAD$, the perpendiculars $OD$, $CF$ are inversely proportional to the sides $AC$, $AD$. Hence, one finds the following expression for the side $CF$.

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158. Cf. Chasles (1837a, pp. 438–441). Not only the sides of the quadrilateral $ABCD$ were rationally expressed in terms of the sides of the triangles $T = (a, b, c)$ and $T' = (a', b', c')$, but also the diagonals and their segments, as well as the perpendiculars. If the sides of the generators are rational numbers, one then obtains a rational quadrilateral.

159. In his notebooks, Chasles derived these formulas by merely applying Brahmagupta’s rule for the height and the segments of a triangle whose sides are known (viz. XII 22) to the side and diagonal of the quadrilateral obtained by another of Brahmagupta’s rules (viz. XII 28).

160. Cf. Chasles (1837a, p. 441). As a matter of fact, the hypotenuse $CD$ was rational in the quadrilateral Chasles considered. The point of the above counterfactual assumption was merely to stress the fact that the expression for the sides $CF$, $FD$ are independent of the hypotenuse $CD$.


162. Suffice it to notice that the area of the triangle $CAD$ can be computed in two ways, either out of the perpendicular $OD$ and the side $AC$, or out of the perpendicular $CF$ and the side $AD$. Hence $\frac{OD \cdot AC}{2} = \frac{CF \cdot AD}{2}$, from which follows $\frac{OD}{CF} = \frac{AD}{AC}$.
As for the other side $FD$, Chasles simply used Brahmagupta’s rule XII 22, which gives the formula for computing the segments determined on the side of a triangle by the perpendicular drawn from the opposite vertex to that side, in terms of the three sides of that triangle,\textsuperscript{163} namely in the case at hand

$$FD = \frac{AD^2 + CD^2 - AC^2}{2AD}.$$  

It is clear from the above expressions, that if both sides $AC$, $AD$ are rational, so are $CF$, $FD$, for $CD$ and $OD$ are assumedly rational. But since $AC = AO + OC$, and $OC$ is also assumed to be rational, it is only required that $AO$, $AD$ be rational. Therefore the whole problem amounts to constructing on the side $OD$, a rectangular triangle $AOD$, whose sides are rational. This can be done, Chasles pointed out, by relying on a formula “much used in the works of Brahmagupta and Bhāscara”,\textsuperscript{164} namely Brahmagupta’s rule XII 35,\textsuperscript{165} for generating rational rectangular triangles, or, as Chasles adjusted it to the current problem:

$$\frac{OD^2}{n} + \frac{1}{4} \left( \frac{OD^2}{n} - n \right)^2 = \frac{1}{4} \left( \frac{OD^2}{n} + n \right)^2,$$

where $n$ is an arbitrary number. Since $OD$ is rational by hypothesis, one takes it as the base of the rectangular triangle, then one makes the upright $AO$ equal to $\frac{1}{2} \left( \frac{OD^2}{n} - n \right)$, and the hypotenuse $AD$ equal to

\textsuperscript{163} Cf. Colebrooke (1817, p. 297): “[XII] 22. The difference of the squares of the sides being divided by the base, the quotient is added to and subtracted from the base: the sum and the remainder, divided by two, are the segments.” Considering a triangle with vertices $A$, $B$, $C$, respectively opposite to the sides $a$, $b$, $c$, the perpendicular $h$ drawn from the vertex $C$ to the opposite side $c$ divides it in two segments $\alpha$ (on the side of $a$) and $\beta$ (on the side of $b$). By a double application of the Pythagorean theorem, one obtains $a^2 = \alpha^2 + h^2$ and $b^2 = \beta^2 + h^2$, hence $b^2 - a^2 = \beta^2 - \alpha^2 = (\beta + \alpha)(\beta - \alpha) = cf$, hence $f = \frac{b^2 - a^2}{c}$. That quantity $f$ which is called the \textit{summit} in Colebrooke’s translation, and the \textit{coraustus} in Chasles’s account, corresponds to the side opposite to the base in a symmetric quadrilateral, obtained by axial symmetry with respect to an axis perpendicular to the base. From $c = \beta + \alpha$, $f = \beta - \alpha$, one eventually infers the expressions in terms of the sides, for the segments $\alpha = \frac{c - f}{2}$ and $\beta = \frac{c + f}{2}$, namely $\alpha = \frac{c^2 + a^2 - b^2}{2c}$ and $\beta = \frac{c^2 + b^2 - a^2}{2c}$.\textsuperscript{164} Cf. Chasles (1837b, p. 47).

\textsuperscript{165} Cf. Colebrooke (1817, pp. 306–307): “[XII] 35. The square of the side assumed at pleasure, being divided and then lessened by an assumed quantity, the half of the remainder is the upright of an oblong tetragon; and this, added to the same assumed quantity, is the diagonal.” The formulas for the three sides of a rectangular triangle, denoted as the base ($b$), the upright ($u$) and the diagonal ($d$), thus depend on two arbitrary parameters: $b = \alpha$, $u = \frac{1}{2} \left( \frac{c^2}{b^2} - \beta \right)$, $d = \frac{1}{2} \left( \frac{c^2}{b^2} + \beta \right)$. For a geometrical interpretation of these formulas in the context of Brahmagupta’s work, see Kusuba (1981, pp. 48–50). By multiplying all three formulas given by Brahmagupta by $2\beta$, one finds back those formulas for Pythagorean triples, known in various traditions, viz. $b = 2\alpha\beta$, $u = \alpha^2 - \beta^2$, $d = \alpha^2 + \beta^2$.\textsuperscript{164}
\[ \frac{1}{2} \left( \frac{OD^2}{n} + n \right), \] both being rational. The rational rectangular triangle \( AOD \) being thus constructed, one will only have to draw the perpendicular \( CF \) from the vertex \( C \) to the opposite side \( AD \), and both perpendicular \( CF \) and segment \( FD \) will be the two sought roots. “And so, [Chasles concluded,] the whole question is solved by a geometrical construction.”

Chasles then showed how one may derive Fibonacci’s formulas, by merely writing down the expressions for both sides \( CF \), \( FD \) in terms of the segments effectively coming into play in that geometrical construction. Since both rectangular triangles \( AOD \) and \( AFC \) are similar, one remarks that

\[
CF = \frac{OD}{AD} \cdot AC, \quad \text{and} \quad AF = \frac{AO}{AD} \cdot AC;
\]

since obviously \( AC = AO + OC \), one then obtains by substitution

\[
CF = \frac{OD \cdot AO + OD \cdot OC}{AD},
\]

\[
AF = \frac{AO^2 + AO \cdot OC}{AD};
\]

but now it is clear that \( FD = AD - AF \), hence by substitution and simplification, one eventually derives the relevant formula for \( FD \)

\[
FD = \frac{OD^2 - AO \cdot OC}{AD}.
\]

Recalling that \( OC = x \), \( OD = y \) are the given roots, and \( CF = x' \), \( FD = y' \) the sought roots, and also that the construction of the rational rectangular triangle \( AOD \) corresponds to the assumption of the auxiliary Pythagorean equation \( a^2 + y^2 = \gamma^2 \), one just has to stipulate that \( AO = a \) and \( AD = \gamma \) to rewrite the above equations in the form

\[
x' = \frac{\alpha y + yx}{\gamma},
\]

\[
y' = \frac{y^2 - \alpha x}{\gamma}.
\]

By shifting to the auxiliary equation \( a^2 + b^2 = c^2 \), with the stipulations \( \frac{a}{\gamma} = \frac{c}{\gamma}, \frac{b}{\gamma} = \frac{a}{c} \), one then obtains Fibonacci’s formulas\(^ {167} \):

\[
x' = \frac{ay + bx}{c},
\]

\[
y' = \frac{by - ax}{c},
\]

which therefore prove to be susceptible of a purely geometrical interpretation.

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\(^{166}\) Cf. Chasles (1837b, p. 48).

\(^{167}\) Chasles’s geometrical interpretation of these formulas requires that only the positive roots are chosen. However, in Brahmagupta’s and Bhāskara’s bhāvanā rules, quantities can be either positive or negative, while in Pacioli’s and Cardano’s procedure, the difference of the partial products is taken positively.
Although Fibonacci’s (or, more faithfully to the sources, Pacioli’s and Cardano’s) formulas are no more than a particular case of Brahmagupta’s (or identically Euler’s) formulas, Chasles claimed that the former “virtually contained”\textsuperscript{168} the latter. By this, he meant that one could “rise, naturally and with no computation whatsoever, from that particular case to the general case”,\textsuperscript{169} that is from the method of solving the particular indeterminate equation $x^2 + y^2 = A$ to the method of solving the general one $Cx^2 + y^2 = A$. For the sake of clarity, we will present synoptically the whole procedure generating a new system of roots $(x', y')$ out of a previously given one $(x, y)$ and the auxiliary equation $a^2 + b^2 = c^2$, by means of Fibonacci’s formulas:

\[
\begin{align*}
    x' &= \frac{ay + bx}{c} \\
    y' &= \frac{by - ax}{c} \\
\end{align*}
\]

\[
\frac{x^2 + y^2}{a^2 + b^2} = \frac{A}{c^2} \quad \Rightarrow \quad \frac{x'^2 + y'^2}{c^2} = A.
\]

Chasles remarked that the method of solving the more general equation $Cx^2 + y^2 = A$ can be immediately obtained therefrom by mere substitution. If one replaces $x$ by $x\sqrt{C}$, $a$ by $a\sqrt{C}$ and $x'$ by $x'\sqrt{C}$, in the conditions, the transformation formulas and the conclusion, the previous procedural pattern happens to be rewritten into the following one:

\[
\begin{align*}
    x' &= \frac{ay + bx}{c} \\
    y' &= \frac{by - Cax}{c} \\
\end{align*}
\]

\[
\frac{Cx^2 + y^2}{Ca^2 + b^2} = \frac{A}{c^2} \quad \Rightarrow \quad \frac{Cx'^2 + y'^2}{c^2} = A.
\]

Chasles then proceeded in two steps. Firstly, he showed that this rewriting process implied shifting the underlying geometry by suitably adjusting it to new requirements. Secondly, he inferred a method of solving the general indeterminate equation $y^2 = Cx^2 \pm A$, from the method of solving the previous equation $Cx^2 + y^2 = A$ – with $C$ positive in both cases – by pointing out that the whole procedural pattern, once found in a geometrical way in the latter case, would boil down to a sequence of mere algebraic identities, which hold whatever the values of the constants $C$ and $A$, hence even when these are assumed negative, so that when the indeterminate equation is of the form

\[
y^2 = Cx^2 \pm A,
\]

the formulas become

\[
\begin{align*}
    x' &= \frac{ay + bx}{c} \\
    y' &= \frac{by + Cax}{c}.
\end{align*}
\]

To complete the first step of his argument and make it fully compelling, Chasles had to establish that the geometrical solution devised for the particular equation, $x^2 + y^2 = A$, could also be applied to the more

\textsuperscript{168} Cf. Chasles (1837b, p. 53).

\textsuperscript{169} Cf. Chasles (1837b, p. 44).
general one, \( Cx^2 + y^2 = A \). The problem amounted to tracing a purely geometrical path from a given system of rational roots, \( Cx^2 + y^2 = A \), to a further one, \( Cx'^2 + y'^2 = A \), with the help of an auxiliary equation also in rational numbers, \( Ca^2 + b^2 = c^2 \). Irrational lines occurring in the process of making the necessary adjustments constituted the main obstacle to merely applying the very same construction as in the particular case. Furthermore, specific considerations regarding the form of the quantities were required. Chasles thus proceeded in this way. Given a system of rational roots \( x, y \) satisfying \( Cx^2 + y^2 = A \), one first assumes a rectangular triangle \( COD \) (see Figure 4), but this time with

\[
OC = x\sqrt{C}, \quad OD = y, \quad \text{and} \quad CD = A
\]

so that the above equation is converted into the Pythagorean property \( OC^2 + OD^2 = A \) for the triangle \( COD \). Then, as previously, one constructs a second rectangular triangle \( CFD \) with the same hypotenuse \( CD \) as \( COD \), namely so that \( CF^2 + FD^2 = CD^2 \). But now, one stipulates

\[
x' = \frac{CF}{\sqrt{C}} \quad \text{and} \quad y' = FD,
\]

so that \( Cx'^2 + y'^2 = A \). However, the root \( x' \) must be rational, hence \( CF \) must be of the form \( p\sqrt{C} \), with \( p \) rational. As previously shown, both rectangular triangles \( AFC \) and \( AOD \) are similar, hence the following proportion holds

\[
\frac{CF}{OD} = \frac{AC}{AD};
\]

therefore, if \( AD \) is rational, and \( AC \) also of the form \( q\sqrt{C} \) with \( q \) rational, then \( CF \) will be of the required form. But now, since

\[
AC = AO + OC = AO + x\sqrt{C},
\]

\( AO \) must be of the form \( r\sqrt{C} \) with \( r \) rational. In contrast to the previous construction, one must now construct upon the rational side \( OD \) a rectangular triangle \( AOD \) whose other side \( AO \) is of the form \( r\sqrt{C} \) with \( r \) rational, and whose hypotenuse \( AD \) is rational. This can be done, Chasles suggested, by molding the well-known formula for the Pythagorean triples

\[
4m^2n^2 + (m^2 - n^2)^2 = (m^2 + n^2)^2,
\]

so as to turn it into another one appropriately codifying the properties required of the rectangular triangle \( AOD \):

\[
\frac{4m^2n^2}{(m^2 - n^2)^2} \cdot OD^2 + OD^2 = \frac{(m^2 + n^2)^2}{(m^2 - n^2)^2} \cdot OD^2.
\]

By setting \( n^2 = C \), one will then have

\[
\frac{4m^2C}{(m^2 - C)^2} \cdot OD^2 + OD^2 = \frac{(m^2 + C)^2}{(m^2 - C)^2} \cdot OD^2,
\]
and so it will suffice to take

\[ OA = \frac{2m}{m^2 - C} \cdot OD \cdot \sqrt{C} \quad \text{and} \quad DA = \frac{m^2 + C}{m^2 - C} \cdot OD. \]

Since \( OA \) and \( DA \) have the required form, and since, as previously shown, the perpendicular \( CF \) is such that

\[ CF = \frac{OD \cdot AO + OD \cdot OC}{AD}, \]

then obviously it will be of the form \( p\sqrt{C} \), with \( p \) rational, whereas the segment \( FD \) will be rational.\(^{170}\)

Therefore, \( \frac{CF}{\sqrt{C}} \) and \( FD \) will be rational, and so will be the sought roots of the equation \( Cx^2 + y^2 = A \) — an equation which in return proves to be solved geometrically. Eventually, as in the case of Fibonacci’s formulas, Euler’s formulas would be obtained by tracing out the expressions of both sides \( CF \) and \( FD \) in terms of the segments contributing to the more general construction.

A last insightful remark by Chasles may highlight the value he himself attached to his conjecture. Ever since Lagrange, Euler was repeatedly held as the first who grasped the significance of the particular problem \( Cx^2 + 1 = y^2 \) for solving the more general indeterminate equation \( Cx^2 \pm A = y^2 \). However, Chasles pointed out, the geometrical solution which he claimed to be Brahmagupta’s method reconstituted, shed light on this very connection in the simplest and clearest way. For it uniquely consisted in the construction of the adjacent rectangular triangle \( AOD \), whose side \( AO \) is of the form \( r\sqrt{C} \), whose other side \( OD = p \) rational, and whose hypotenuse \( AD = q \) also rational — a geometrical problem which, in indeterminate analysis, would amount to solving the equation \( C \cdot r^2 + p^2 = q^2 \), or \( C(\frac{r}{p})^2 + 1 = (\frac{q}{p})^2 \), hence \( Cx^2 + 1 = y^2 \).

“The geometrical considerations illuminate [Chasles purposely emphasized as a final argument.] how this auxiliary equation enters into the question, and therein plays the important role Brahmagupta and Euler acknowledged.”\(^{171}\)

5. A few manuscripts and an adjustment

In the second volume of his Histoire des sciences mathématiques en Italie (1838), Libri provided new evidence on Fibonacci and pronounced a rather unfavorable judgment on Chasles’s interpretation about the import of the Liber quadratorum for the overall history of indeterminate analysis. In an appendix to his book, he edited parts of Fibonacci’s Liber Abaci on the basis of a fourteenth-century manuscript he found in the Magliabechian library in Florence. He offered the transcription of the well-known incipit,\(^{172}\) as well as of the whole chapter XV, which he presented as Fibonacci’s “complete treatise of algebra.”\(^{173}\)

Libri’s main point was to credit Fibonacci for being the first who introduced algebra among the Christians. As regards the Liber quadratorum, he looked in vain for the lost manuscript, which was still extant in

\(^{170}\) That \( FD \) must be rational is easily seen, whether one refers to one formula or the other, both previously seen, viz. either

\[ FD = \frac{OD^2 - AO \cdot OC}{AD} \] (for \( AO \) and \( OC \) are both of the form \( r\sqrt{C} \)), or \( FD = \frac{AD^2 + CD^2 - AC^2}{2AD} \) (for the only quantities of the form \( r\sqrt{C} \) occur in squares).

\(^{171}\) Cf. Chasles (1837b, p. 55).


1768, when Targioni consulted it. In contrast to Cossali, who apparently did not know about the early sixteenth-century author Francesco Ghaligai, Libri claimed that one should be able to restore the content of Fibonacci’s lost work, better than had been done previously, by closely comparing Pacioli’s and Ghaligai’s accounts of it. However, he did not carry out this task, except for a few sweeping statements. He did not even mention the solution to the equation $x^2 + y^2 = A$ among Fibonacci’s original contributions. Beside his strong disagreement with Chasles over the origin of the positional numeral system, Libri also pointed out his rival’s purported failure to correctly assess the originality of Fibonacci’s work in indeterminate analysis. He denied that Fibonacci may have borrowed his formulas for solving $x^2 + y^2 = A$, from the Arabs, who in return would have inherited these from Brahmagupta, as Chasles had contended, allegedly at the cost of inconsistency. Indeed, Libri argued, “when [on the one hand] Fibonacci says that he took over the Hindu numeral system from the Arabs, M. Chasles wants to prove that this system is occidental; and when [on the other hand,] the geometry of Pisa says that he wrote on square numbers after some questions were proposed to him by some philosophers from the court of Frederick II, M. Chasles claims that he borrowed his researches from the Arabs, although all the ancient geometers who wrote on these matters quote Leonard without ever quoting any Arabic work (which they always do when it comes to the solution of determinate equations of the second degree, and to what they call algebra), and although no Arabic work had ever been found, in which relatively higher issues of indeterminate analysis were dealt with.” In Libri’s view, both Chasles’s interpretations of Boethius’ passage of the Pythagorean abacus and of Brahmagupta’s theory of quadrilaterals were denounced as equally adventurous.

In historical matters, one should guard against mistaking one’s suppositions for realities, and drawing consequences therefrom. Here [viz. as regards Fibonacci taking over Brahmagupta’s rules.] M. Chasles seems to have granted too much credit to his own hypotheses. . . . Interpretations are always dangerous: the interpreter’s talent is often a substitute for the author’s, and M. Chasles has probably been very generous to Brahmagupta and Boethius, when, owing to slight changes in the text, he attributed to them results they maybe never knew.

In Chasles’s view however, Libri’s skepticism was rooted in the ungrounded assumption that, since no Arabic work on higher topics of indeterminate analysis was yet known, no such work would ever be. On the contrary, Chasles constantly encouraged historical research in the field of Arabic mathematics. “Today the works of the Arabs of which we know a too small number [he pointed out,] and to which too scant attention has been paid because one believed one would find there only a feeble echo of the Greek

174 Libri invited his fellow scholars interested in glorifying Italy to look for that precious manuscript, cf. Libri (1838–1841, II, p. 27): “Nous engageons tous ceux qui s’intéressent à la gloire de l’Italie de rechercher ce précieux manuscrit: il ne peut qu’être égaré, et celui qui le retrouvera aura bien mérité des sciences.” Boncompagni later fulfilled Libri’s wish.

175 Libri criticized Cossali for dressing Fibonacci’s rules “in a garb slightly too modern”, while mixing them with Euler’s, Lagrange’s, which presumably made his account useless for historical research. Cf. Libri (1838–1841, II, pp. 41–42).


177 Cf. Libri (1838–1841, II, p. 42): “Pour indiquer quelques recherches originales de Fibonacci, nous dirons qu’il donna: (1) la somme de la série des nombres naturels et des nombres carrés, (2) la formule générale pour former les triangles arithmétiques, et (3) la résolution particulière de ce problème difficile: trouver un carré auquel, en ajoutant ou en soustrayant un nombre donné, on ait toujours un carré.”


school, must inspire more interest, now that one recognizes there the pronounced traces of another source of enlightenment [viz. the Indian one].

By taking this stance, Chasles exerted his influence to stimulate and support the works of younger scholars such as Louis-Amélie Sédillot (1808–1875) and Franz Woepcke (1826–1864). By the time Chasles published his Aperçu historique, the younger Sédillot, a protégé of Silvestre de Sacy, had already launched into a lifelong research program in defense of the originality of the Arabs in the exact sciences, a program his father Jean-Jacques Sédillot (1777–1832) had first adumbrated. In 1834, Louis-Amélie Sédillot published a paper in the Journal asiatique, in which he incidentally mentioned a fragment of an algebraic treatise, found among six other mathematical works contained in the Arabic manuscript n° 1104 of the Bibliothèque royale. This fragment revealed that, contrary to what was then believed, the Arabs had successfully dealt with determinate equations of the third degree. Chasles immediately saw the importance of this discovery, all the more so as it provided unexpected grist to his own mill. “This work is a fragment of algebra, ... in which the equations of the third degree are solved geometrically”. Chasles noted with emphasis, before paying tribute to the forthcoming edition of that work, then announced but never achieved. In 1838 however, Sédillot offered a detailed analysis of the content of the fragment and an outline of the geometrical methods for solving cubic equations by the intersection of two conics. In 1851, Franz Woepcke identified the author of that mysterious fragment, and, on the basis of two other manuscripts in addition to the one Sédillot had found, he published the editio princeps of what he eventually established as the Algebra of Al-Khayyām (1048–1131).

Still, as Woepcke later indicated, there remained a gap to fill with regard to indeterminate analysis among the Arabs. Sources were missing to settle this issue with historical accuracy. Woepcke then stumbled upon a hitherto unnoticed manuscript brought back from the Napoleonic expedition to Egypt, and kept in the collections of the Bibliothèque impériale. In 1853, he translated into French significant excerpts of an eleventh-century algebraic treatise by al-Karajī (c. 953–c. 1029), known as the Fakhrī, which shed light on the disputed question of the contribution of the Arabs to indeterminate analysis. Indeed, Woepcke claimed that this new source enabled him to prove the five following points, namely:

1. That the Arabs knew indeterminate analysis;
2. That their works on this topic are based upon the work of Diophantus;
3. That by inventing new procedures, as well as by considering problems of higher degrees, they added to Diophantus’s algebra;
4. That till the end of the Xth century, they ignored the methods of indeterminate analysis found among the Indians;

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180 Cf. Chasles (1837b, p. 45).
181 On the work of Louis-Amélie Sédillot and the controversy of the lunar variation, see Charette (1995, Chapter 5).
183 Cf. Sédillot (1834, pp. 435–436): “Le plus important de ces opuscules est un fragment d’un traité d’algèbre qui prouve que les Arabes connaissaient les équations cubiques, et qui résout heureusement cette question, encore controversée dans l’histoire de la science. L’auteur de cet ouvrage ne se nomme point, mais comme il le dédie à un grand juge ou chancelier, il nous sera peut-être possible d’avoir la date approchée de sa composition.”
184 Cf. Chasles (1837a, p. 493).
185 Cf. Chasles (1837a, p. 498, note 3): “M. Sédillot promet de faire connaître ces pièces, dont une, qui est le fragment d’algèbre sur la résolution des équations du troisième degré dont nous avons parlé plus haut sera l’un des monuments les plus précieux de l’histoire des sciences chez les Arabes.”
187 Cf. Woepcke (1851).
5. That Fibonacci’s works do not have the degree of originality one has tried to attribute to them, but that they are in great part borrowed from the Arabs, and in particular from al-Karaji.\textsuperscript{188}

As regards the \textit{Liber quadratorum}, then deemed lost, Woepcke warned that he had to rely on Cossali’s account so as to compare Fibonacci’s methods to those attested in al-Karaji’s treatise. And hence his conclusions were submitted “as conjectures, rather than as the results of a rigorous scrutiny”.\textsuperscript{189} Although Fibonacci’s notion of square numbers as sums of successive odd numbers proved alien to al-Karaji, the borrowings of the former from the latter were presented as unquestionable when it comes to the proposed problems.\textsuperscript{190} Woepcke then compared the \textit{Fakhri} with Brahmagupta’s and Bhaskara’s works as translated by Colebrooke. “The result of this examination was negative, with respect to the general character of the methods, as well as to the particular elements composing the Arabic treatise and the Indian ones.”\textsuperscript{191} Woepcke’s verdict was final. “In al-Karaji, one neither finds the method of the Indians for solving indeterminate equations of the first degree in integers (which is essentially in line with the modern one), nor the method, for finding an infinite number of solutions to the equation $cx^2 + a = y^2$ (which was discovered a second time by Euler), nor any of the other Indian methods for solving indeterminate equations of the second degree.”\textsuperscript{192} Woepcke then concluded that by the end of the tenth century, the influence of the Greeks was the only noticeable one in Arabic indeterminate analysis. However, he felt compelled to “say a word about an objection that naturally comes to the mind of the reader”, \textit{viz.} Chasles’s conjecture.

M. Chasles has shown, by making ingenious connections, that Fibonacci’s solution to the indeterminate equation $x^2 + y^2 = a^2 + b^2$ is essentially contained in a geometrical theorem by Brahmagupta; \ldots Yet, Fibonacci could only know of Indian methods through the Arabic mathematicians. Therefore, if the above mentioned solutions are to be necessarily derived from Brahmagupta’s theories, one is forced to bestow upon the Arabs a knowledge of Indian methods which I cannot find in \textit{al-Karaji}.\textsuperscript{193}

Out of respect for his mentor, Woepcke nevertheless suggested a way to save Chasles’s conjecture. One might have supposed indeed that the Arabic authors only became aware of Indian algebra within the time span of two centuries between al-Karaji and Fibonacci. However, Woepcke warned, one would “in vain dispel a difficulty by a hypothesis”,\textsuperscript{194} and only further historical research would yield a definitive answer.

The following year, in 1854, Baldassare Boncompagni (1821–1894)\textsuperscript{195} edited the \textit{Liber quadratorum},\textsuperscript{196} as well as the \textit{Flos}, a collection of problems hitherto completely unknown. He had discovered

\textsuperscript{188} Cf. Woepcke (1853, p. 3). For the sake of clarity, current transliteration is favored for Arabic names. Woepcke writes \textit{Alkarkhî} for \textit{al-Karaji}.

\textsuperscript{189} Cf. Woepcke (1853, p. 32).

\textsuperscript{190} For a comprehensive analysis of the relationship between Fibonacci and Arabic mathematics, one may refer to the work of Roshdi Rashed, cf. Rashed (2011, Part I, section II, chapters 4 and 5, pp. 347–379). Fibonacci’s \textit{Liber quadratorum} is presented there as a Latin extension of the tradition of Arabic mathematics, albeit of the first period (IXth–Xth centuries) only. This perspective induces a corrective to some of Woepcke’s views, cf. Rashed (2011, p. 362): “Notons que les problèmes qui, selon Woepcke, auraient été empruntés par Fibonacci au mathématicien de la fin du Xème siècle, al-Karaji, ou à Diophante \textit{via} ce dernier, se trouvent tous dans le livre d’Abû Kâmil. Or rien n’indique, contrairement à ce que croyait Woepcke, que Fibonacci connaissait le livre d’al-Karaji – \textit{al-Fakhri} – non plus que les \textit{Arithmétiques} de Diophante.” As regards the algebraic problems dealt with in Fibonacci’s \textit{Liber Abaci}, an overview of the borrowings from Abû Kâmil’s Algebra is provided in (Rashed, 2012, pp. 17–26).

\textsuperscript{191} Cf. Woepcke (1853, p. 32).

\textsuperscript{192} Cf. Woepcke (1853, p. 34).

\textsuperscript{193} Cf. Woepcke (1853, p. 43).

\textsuperscript{194} \textit{Ibid.}


\textsuperscript{196} Cf. Boncompagni (1854, pp. 55–122), and also Boncompagni (1857–1862, II, pp. 253–283). The 1854 edition of Fibonacci’s works drew the attention of the Italian mathematician Angelo Genocchi (1817–1889) who, out of interest for the history of number
both manuscripts in a codex kept at the Ambrosiana library in Milan. With the text of Fibonacci’s *Liber quadratorum* being eventually available, it became possible to assess the correctness of Chasles’s interpretation. More specifically, one could check the original to see what may have matched Pacioli’s intimation, which Chasles found so suggestive, that Fibonacci’s formulas were proved “by the consideration of geometrical figures”. As a matter of fact, these formulas had been obtained by concatenating two propositions of the *Liber quadratorum*, provided the expressions were duly transcribed in algebraic notation. Proposition 6 [in Sigler’s translation]\(^{197}\) states that, given \(a, b, e\) and \(f\) four numbers such that \(a < b\), \(e < f\) and \(\frac{e}{f} \neq \frac{a}{b}\), the product \((a^2 + b^2)(e^2 + f^2)\) is equal to the sum of two squares in two, three or four ways. Fibonacci’s proof essentially rests upon the algebraic identities
\[
(a^2 + b^2)(e^2 + f^2) = (ae \pm bf)^2 + (be \mp af)^2,
\]
which in return are established by means of arguments with line segments in the style of Euclidean geometrical algebra.\(^{198}\) Proposition 9 then makes use of the above to find two numbers which have the sum of their squares equal to a non square number which is itself the sum of the squares of two given numbers.\(^{199}\) Assuming a non square number \(A\) equal to the sum of two squares \(x^2 + y^2\), and a Pythagorean triple \(a^2 + b^2 = c^2\), Fibonacci forms the product \(c^2 A\), and applies the above algebraic identities in various ways. The formulas which Chasles attributes to Fibonacci are then obtained by considering one of those identities
\[
(a^2 + b^2)(x^2 + y^2) = (ay + bx)^2 + (by – ax)^2,
\]
from which the solution immediately follows
\[
x' = \frac{ay + bx}{c}, \quad \text{and} \quad y' = \frac{by – ax}{c}.
\]

Here, Fibonacci’s proof essentially relied on the use of geometrical figures, namely similar triangles.\(^{200}\) Together with the Euclidean-type geometrical algebra arguments, this latter feature most probably corresponds to what Pacioli had in mind when he characterized Fibonacci’s method, but obviously it does not perfectly fit with Chasles’s expectations.

In a paper published in the Liouville journal in February 1855, Woepcke analyzed Boncompagni’s publication. As a result, he slightly modified his previous conclusions, in particular regarding the originality of


\(^{198}\) On Fibonacci’s use of Euclidean-type geometrical algebra, cf. Sigler (1987, p. xi): “The geometrical algebra used by Leonardo is that presented by Euclid in the *Elements*: Leonardo had great facility with geometrical algebra. . . . It is easy to be impressed by the elegance and the intricacy of his arguments in a geometrical notation that certainly did not facilitate algebraic insight. Yet Leonardo’s mastery and dexterity with the crude algebra were astonishing.” On Fibonacci’s proof of the above algebraic identities, see also Rashed (2013, p. 112): “La démonstration de Fibonacci est dans le plus pur style des livres arithmétiques des *Éléments*.”


\(^{200}\) On Fibonacci’s arguments based on the similarity of triangles in the *Liber quadratorum*, cf. Rashed (2011, p. 368): “À l’évidence, la méthode de Fibonacci n’est ni celle de Diophante dans ses *Arithmétiques*, ni celle, algébrique, d’un Abū Kämîl par exemple. . . . sa méthode est géométrico-arithmétique. La géométrie est présente en personne, dans la mesure par exemple où Fibonacci a recours à la similitude des triangles. Mais cette méthode n’est pas non plus celle des théoriciens de la nouvelle analyse diophantienne, comme Al-Khâzin ou al-Sjîzî.”
Fibonacci’s methods in the *Liber quadratorum*. While emphasizing the contrast between Diophantus’s and Fibonacci’s approaches to indeterminate analysis, he simply highlighted the noteworthy topics dealt with in the Latin treatise, and, among them, the above mentioned propositions [*viz. 6 and 9*]. However, this time, he did not even mention Chasles’s conjecture. Shortly afterwards, on April 2, 1855, Chasles made a communication before the Paris *Académie des sciences*, in which he presented recent comments and replies prompted by Boncompagni’s edition of Fibonacci. Firstly, he read a letter from Genocchi to Jean-Baptiste Biot, dated March 21, 1855, which enumerated Fibonacci’s most significant results from the standpoint of the history of science. He also communicated a letter from Boncompagni to himself, dated March 19 of that same year. Boncompagni drew his attention to Fibonacci’s solution of the indeterminate equation $x^2 + y^2 = A$, and acknowledged the relevance of at least part of Chasles’s conjecture. “Fibonacci’s rule, [Boncompagni remarked,] which he applies to two numerical examples, coincides with the formulas Luca of Burgo and Cardano had given without demonstration, and the importance of which you made known in your Note on the indeterminate equations of the second degree.” Interestingly enough, Chasles’s response to Boncompagni, inserted in a footnote, reveals how he took into account the newly available evidence to adjust his conjecture, far from renouncing it.

This Note shows that Fibonacci’s formulas, reported by Luca of Burgo and Cardano, coincide with those virtually contained in a question of geometry dealt with by Brahmagupta. It makes clear that one can deduce from these simple formulas, by means of a geometrical construction, the general ones pertaining to the solution of the equation $Cx^2 \pm A = y^2$, which one found with astonishment and admiration in the Hindu works. It is not improbable that this geometrical way may have been the way by which the Hindu geometers were led to these so remarkable formulas, which would suffice to attest to the character of originality and the high scientific value of the mathematical speculations of that ancient people. These formulas, stamped with the degree of elegance and perfection to which the topic was liable, differ, as one knows, from Diophantus’s formulas, and they remained unknown to modern geometers, who, for one and a half centuries mostly, dealt with indeterminate analysis, until the middle of the last century when Euler gave them under the form in which they occurred, without his knowledge, for long centuries in the works of the Hindus.

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201 Cf. Woepcke (1855, p. 55): “Le moyen principal et presque unique employé par Fibonacci pour résoudre les problèmes indéterminés du second degré dont il s’occupe, est tout à fait original, et ne se trouve ni chez Diophante, ni chez les Indiens, ni chez les Arabes. C’est une étude très-approfondie des suites formées par la succession des carrés des nombres naturels et des nombres impairs, ainsi que des différences de ces suites.”

202 Ibid. “[Diophante] ne se préoccupe que d’obtenir des solutions rationnelles, tandis que [Fibonacci] résout la plupart de ses problèmes en nombres naturels.”

203 About the above algebraic identities, Woepcke stressed Fibonacci’s merit by recalling that Cauchy was not averse to providing a special proof for them in his *Cours d’Analyse*, “comme un exemple de la grande utilité des expressions imaginaires, ‘non seulement dans l’algèbre ordinaire, mais encore dans la théorie des nombres’” (Cauchy), cf. Woepcke (1855, p. 57).

204 About Fibonacci’s use of the above algebraic identities for proving various theorems on the number of decompositions of a whole number into two squares, Genocchi simply remarked, without further comment, that “all these propositions are followed by a demonstration in which [Fibonacci] helps himself with a geometrical figure”, cf. Genocchi (1855, p. 776).

205 Cf. Chasles (1855, p. 780). 

206 Cf. Chasles (1855, p. 780): “On montre dans cette Note que les formules de Fibonacci, rapportées par Lucas de Burgo et Cardan, coïncident avec celles que renferme virtuellement une question de Géométrie traitée par Brahmagupta, et l’on fait voir qu’on peut déduire de ces simples formules, au moyen d’une construction géométrique, les formules générales relatives à la résolution de l’équation $Cx^2 \pm A = y^2$, qu’on a trouvées avec étonnement et admiration dans les ouvrages hindous. Il n’est pas improbable que cette voie géométrique ait été celle qui a conduit les géomètres hindous à ces formules si remarquables, qui suffiraient pour attester le caractère d’originalité et la haute valeur scientifique des spéculations mathématiques de cet ancien peuple. Ces formules, empreintes du degré d’élégance et de perfection dont le sujet était susceptible, sont différentes, comme on sait, de celles de Diophante, et sont restées inconnues aux géomètres modernes, qui depuis un siècle et demi surtout s’occupaient de l’analyse indéterminée, jusqu’au milieu du siècle dernier où Euler les a données sous la forme même où elles se trouvaient, à son insu, depuis de long siècles dans les ouvrages des Indiens.”
Chasles thus unwaveringly maintained his views about some formulas akin to those of Fibonacci being “virtually contained” in Brahmagupta’s geometrical theory of quadrilaterals, a notion he reiterated almost unchanged, except for his previous bold assumption – which he henceforth tacitly jettisoned – of a historical connection of some kind, presumably through the Arabs, between Indian mathematics on the one hand, and Fibonacci and European medieval mathematics on the other hand. On this last score, he took his cue from Woepcke, whose conclusions he fully endorsed, as Terquem also did in his overall account of Boncompagni’s work and its reception. In his analytical notes on the Liber quadratorum, Genocchi compared Fibonacci’s formulas with Diophantus’s alternative solution and suggestively concluded by emphasizing again Chasles’s point: “The more elegant formulas of Leonardo Pisano can be drawn from a geometrical question dealt with by the Indian mathematician Brahmagupta, as shown by M. Chasles.”

As a mere interpretive assumption about the original way algebra and geometry might have been connected in Sanskrit sources, Chasles’s conjecture therefore continued to be endowed with an aura of authority.

Less than two decades later, however, in his notes, posthumously gathered into a book Zur Geschichte der Mathematik in Altertum und Mittelalter (1874), Hermann Hankel cast a fresh eye on the same Sanskrit sources Chasles had sought to interpret within a unified geometrical framework. As Chasles before him, he was also dependent on Colebrooke’s translations for his own assessment of the mathematical content presumably articulated therein. While carefully reconsidering both Brahmagupta’s theory of quadrilaterals and the whole set of Sanskrit methods of indeterminate analysis from Brahmagupta’s bhāvanā rules to Bhāskara’s cakravāla, Hankel did not connect these two pieces of mathematics in the straightforward way Chasles had envisioned. As a matter of fact, there is no hint that he ever explored any such connections, at least not in the notes he left unfinished and which were published in his name after his death. Furthermore his own contributions on both scores should not be deemed on a par. Whereas Hankel reinterpreted Brahmagupta’s theory of quadrilaterals in a thoroughly original and perceptive way, his comments on Sanskrit indeterminate analysis, however astute, do not bear the same stamp of novelty. Although he had a perfect command of all the intricacies of Chasles’s so-called restoration of Brahmagupta’s geometry, Hankel did not follow the French geometer’s lead when it came to indeterminate analysis, for, as a counterweight, he also greatly benefited from the later expertise of such scholars as Woepcke and Boncompagni.

From Woepcke, he took for granted “the fact of great historical interest that, in al-Karaji’s great work [viz. the Fakhri], one does not find the least trace of any acquaintance with the much more accomplished indeterminate analysis of the Indians.” However, while heavily drawing on their work, Hankel did not embrace Woepcke’s and Sédillot’s claim regarding the purported originality of the Arabs. In his view, the latter only investigated further what they had received from other peoples, however aptly, but did not con-

\[\text{207} \text{ In his 1855 address to the Académie des sciences, Chasles presented Woepcke’s work on al-Karaji’s Fakhri, and he mentioned Woepcke’s detailed comparison between the Liber Quadratorum and Indian, Greek as well as Arabic traditions in indeterminate analysis, cf. Chasles (1855, pp. 782–783): “Cette analyse comparative est d’un grand intérêt; elle éclaire enfin un point d’histoire des sciences mathématiques qui restait couvert d’obscurité. On savait seulement, par les biographies et quelques catalogues de manuscrits orientaux, que les Arabes avaient connu et commenté l’ouvrage de Diophante, mais on ignorait ce qu’ils avaient écrit sur cette partie de l’algèbre, et jusqu’à quel point ils avaient pu y faire des progrès. M. Woepcke conclut que l’analyste arabe Al-Karkhi a souvent reproduit fidèlement l’ouvrage de Diophante, qu’il ne paraît pas avoir connu les ouvrages hindous, et que Fibonacci paraît avoir connu l’ouvrage d’Al-Karkhi.”}\]

\[\text{208} \text{ Cf. Terquem (1855–1856, vol. 15, pp. 61–62). Terquem remained silent there about Chasles’s conjecture.}\]

\[\text{209} \text{ Cf. Genocchi (1855, p. 57): “Le formule più elegante di Leonardo Pisano si possono trarre, come mostrò il signor Chasles, da una questione geometrica che trattò il matematico indiano Brahmagupta.”}\]

\[\text{210} \text{In his interpretation of Brahmagupta’s theory of quadrilaterals, Hankel cunningly drew on the previous interpretive attempts made by both Chasles and the German number theorist Ernst Kummer, who himself creatively responded to Chasles’s reading. Hankel exploited the respective strengths of both his predecessors so as to shape his own interpretation, presumably more in step with philological standards. The whole sequence will be analyzed in a forthcoming publication.}\]

\[\text{211} \text{ Cf. Hankel (1874, p. 270).}\]
tribute on their own anything comparable to such achievements of ancient mathematics as "the theory of conics and curves of the Greeks and the indeterminate analysis of the Indians".\textsuperscript{212} Suggestively enough, he for instance considered Baghdad as a "neutral medium in which both poles would be integrated so that all that the Greeks and the Indians had achieved may be transmitted to the European peoples."\textsuperscript{213} A number of historical and social factors pertaining to the German context may probably account for his being more responsive to the Sanskrit sources than to the Arabic ones, in contrast to Woepcke who was much more connected to the Paris orientalist milieus.\textsuperscript{214} As regards Fibonacci, Hankel analyzed the methods of the\textit{ Liber quadratorum} on the basis of Boncompagni's edition.\textsuperscript{215} He also appropriated Woepcke's second thoughts on Fibonacci's originality with respect to his Arabic sources. Concerning Indian contributions, he registered that the method for solving linear indeterminate equations is to be found first in Āryabhaṭa in the fifth century already, although, he added, to all appearances the material was only methodically arranged by Brahmagupta in the seventh century.\textsuperscript{216} Besides, he checked with the Leiden Sanskritist Hendrik Kern (1833–1917) that, contrary to Colebrooke's suggestion, "the solution to\( ay^2 + 1 = x^2\) does not occur in the Āryabhaṭiya".\textsuperscript{217}

With respect to the interpretation of the cakrāvāla, Hankel barely added to Strachey's account, except for explicitly pointing out that, however praiseworthy this method may be, it lacked a proof of correctness in addition to the proof that it reaches its goal in any case.\textsuperscript{218} While definitely putting strong emphasis on the latter issue, Strachey apparently sidestepped the former. Although Hankel never quoted expressly from Strachey,\textsuperscript{219} he could not ignore his contribution on Bhāskara's\textit{ Bīja-ganita}, for both Colebrooke and Chasles referred to it. Beside the British Indologists, Hankel also drew on the work of the German mathematician Arthur Arneth (1802–1858),\textsuperscript{220} who had commented in great detail upon the Sanskrit methods of indeterminate analysis he came to know about through Colebrooke. Although he acknowledged that "the cyclic method ... leads easily and quickly to its goal", Arneth noted that "it leaves much to the discretion of the calculator, but Bhāskara also remarks: algebra is sagacity [Scharfsinn], the calculator must know how to help himself".\textsuperscript{221} Still, Arneth only carefully described the procedure, but he did not venture to explain its rationale, nor did he refer in the least to Lagrange for this. Hankel made the best of all this material to provide a clear account of the "cyclic method",\textsuperscript{222} by supplying the proof of correctness and the solution to the halting problem, which, he claimed, were missing in the Sanskrit sources.

Regarding the proof of correctness, Hankel established that the quantities yielded by the cakrāvāla procedure are integers (see Figure 5), by merely relying on divisibility properties. His main insight was probably

\textsuperscript{212} Cf. Hankel (1874, p. 226).
\textsuperscript{213} Cf. Hankel (1874, p. 227).
\textsuperscript{214} Some of these historical and social factors are investigated in another publication, cf. Smadja (2015).
\textsuperscript{215} On Hankel's account of Fibonacci's\textit{ Liber quadratorum}, see Hankel (1874, pp. 346–347).
\textsuperscript{216} This method for solving equations of the form \(ax + by = c\) "was worked out into an algorithm called the 'pulverizer' [Zerstäubung] by the Brahmins, an algorithm [Hankel pointed out] which does not distinguish itself from the development of \(\frac{a}{b}\) in a continued fraction, to which, since Euler, any solution amounts", see Hankel (1874, p. 197).
\textsuperscript{217} Cf. Hankel (1874, p. 203). Hankel relied here on a written communication by Hendrik Kern, who was then preparing the edition of the Āryabhaṭiya, viz. (Kern, 1874).
\textsuperscript{218} Cf. Hankel (1874, p. 203): "In der Tat fehlt der 'cyklischen' Methode der Brahmnen nichts, als zunächst der Beweis ihrer Richtigkeit, ... und ferner der Nachweis, dass sie ... zum Ziele führt." For this reason, Hankel proposed to rename Pell's equation the "Indian equation".
\textsuperscript{219} In his later account, Konen closely followed Hankel, and therefore failed to do justice to Strachey, while giving all the credit of the interpretation Strachey first articulated to Hankel, cf. Konen (1901, pp. 18–28).
\textsuperscript{220} On Arneth's history of mathematics and its role in shaping Hankel's reading of Colebrooke, see Smadja (2015).
\textsuperscript{221} Cf. Arneth (1852, pp. 162–163). Arneth refers here to Colebrooke's translations, see Colebrooke (1817, p. 276): "It is apparent to men of clear understanding, that the rule of three constitutes arithmetic, and sagacity, algebra."
\textsuperscript{222} On Hankel's explanation of the cakrāvāla, see Hankel (1874, pp. 200–203).
The positive bhāvanā rules

Starting with a given equation
\[ Na^2 + k = b^2, \]
one chooses the closest square \( m^2 \) to \( N \), then one applies the positive bhāvanā rules to both sets of values:

<table>
<thead>
<tr>
<th>Least Root</th>
<th>Greatest Root</th>
<th>Interpolator</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{am+b}{k} )</td>
<td>( \frac{bm+Na}{k} )</td>
<td>( k(m^2-N) )</td>
</tr>
</tbody>
</table>

One determines the integer \( m \) so that \( a' = \frac{am+b}{k} \)
is an integer, and \(|r^2-a|\), also an integer, is the smallest possible. Then one sets

\[
\begin{align*}
    b' &= \frac{bm+Na}{k} \\
    k' &= \frac{m^2-N}{k}
\end{align*}
\]

Why should these quantities be integers? Mere divisibility properties suffice to prove that this is so and further that the equation \( Na'^2 + k' = b'^2 \)
holds.

The negative bhāvanā rules

Starting with a given equation
\[ Na^2 + k = b^2, \]
one posits another equation
\[ Na'^2 + k' = b'^2, \]
so that, by applying the negative bhāvanā rules, one obtains:

<table>
<thead>
<tr>
<th>Least Root</th>
<th>Greatest Root</th>
<th>Interpolator</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a' )</td>
<td>( b' )</td>
<td>( k' )</td>
</tr>
<tr>
<td>( \frac{a'b-ab'}{k} )</td>
<td>( \frac{bb'-Naa'}{k} )</td>
<td>( kk' )</td>
</tr>
</tbody>
</table>

One then determines the integers \( a' \) and \( b' \), so that \( a'b-ab'=1 \). One also sets \( m = bb' - Naa' \), so that \( N.1^2 + kk' = m^2 \), therefore \( m^2 - N \) is divisible by \( k \), and \( k' = \frac{m^2-N}{k} \) must be an integer. Now from both assumptions

\[ a'b-ab'=1, \quad bb'-Naa'=m \]

one easily derives

\[ a' = \frac{am+b}{k}, \quad b' = \frac{a'b-1}{a}. \]

Figure 5. Hankel’s proof of correctness.

to suggest how one could figure out “the train of thought by which the Indians may have come to this method”, 223 by considering opposite instances of the bhāvanā rules running in parallel, viz. the so-called positive and negative bhāvanā rules (respectively left and right columns in Figure 5), each instance shedding light on the other by making clear how the quantities occurring on one side as operands also occur on the other side as results, and vice versa. By their mutual interlocking, both instances of the bhāvanā rules would presumably complement each other, and, as a whole, make a consistent computational proof.

As for the solution to the halting problem, Hankel claimed, as Strachey before him, that only Lagrange’s theory would eventually yield it. On this last score, Hankel therefore came much closer to Strachey than to Colebrooke, who never took into account such technicalities and did not even mention continued fractions. However, unlike Strachey who more cautiously acknowledged the analogies between Sanskrit and modern mathematics as being merely instrumental, Hankel did not refrain from simply identifying the cakravāla with the Euler–Lagrange method. 224

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223 Cf. Hankel (1874, p. 201).
224 Cf. Hankel (1874, p. 202): “[The cakravāla] is in a remarkable way exactly the same method as the method Lagrange presented in a memoir published in 1769, and only later reduced to the continued fraction algorithm for \( \sqrt{a} \), which Euler had applied to this problem in 1767.”
6. Epilogue

Although neither Strachey, nor Chasles, nor Hankel, had access to the Sanskrit text of Bhāskara and Brahmagupta, their mathematically oriented interpretation of these sources – which therefore they only addressed in translation – may arguably help us to grasp significant aspects of the Sanskrit methods of indeterminate analysis in discussion, and thus prove useful to future historians complying with more stringent philological standards. Let us point out, in conclusion, those parts of their respective contributions which, sieved through critical hindsight, may be considered somehow as standing the test of time as valuable ingredients for further historiographical inquiries. Strachey’s reading of Bhāskara’s Bhāgya-ganita in the light of Lagrange’s self-centered historiography of indeterminate analysis decisively opened the way to an understanding of the Sanskrit methods by providing a systematic overview of their interrelationship. More specifically, his nuanced reliance on Lagrange’s theory of periodic continued fraction expansions for quadratic surds so as to account for both the cyclicity of the cakravāla and the corresponding halting problem, stands out as a fine interpretive stance. In contrast to later scholars, Strachey used Lagrange’s mathematics to shed light on Sanskrit methods, but always resisted any fictitious straightforward identification. Apart from his tentative suggestion of an uninterrupted tradition from Sanskrit to modern mathematics, through Fibonacci and the Arabs, which later scholarship irretrievably condemned, Chasles interestingly highlighted possible connections between algebraic procedures and geometrical constructions in the case of indeterminate equations of the second degree. Although his point about the geometry plausibly underlying the algebraic procedures may be convincing for the equation $x^2 + y^2 = A$ – from which the whole notion of a correspondence originated – and consequently also for $Cx^2 + y^2 = A$, still, one may have reasonable doubts about the last step, viz. the extension to the general case $y^2 = Cx^2 ± A$, being made “naturally and with no computation whatsoever”, for, to all appearances, in that instance, geometry no more leads the way. Eventually, Hankel usefully provided a proof of correctness, hitherto missing in previous accounts of the cakravāla, by investigating the interlocking symmetry between the positive and the negative bhāvanā rules. How these interpretive nuggets may connect to one another remains an open question.

By breaking the well-polished “superstructural” narratives of past historiographies into the particulars of their underlying “infrastructural” groundwork, one may perhaps take advantage of their respective strengths without though endorsing their weaknesses. While disentangling the former from the latter, one may even hopefully form a toolkit of short-ranged, but effective, interpretive devices, which, independently of the grand historiographic schemes in which they were first encased, may provisionally afford functional explanatory modules well-adjusted to the available evidence. Further historiographical attempts to grapple with the sources might then benefit from this critical inventory of past endeavors, by freely recombining, as the case may be, these limited modules in new ways presumably more in step with current concerns and standards. By intertwining these various threads, untied from their original plots, critical history of historiography may also contribute to shape the historiography of the field as a cumulative enterprise.

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